Quaternion Least Mean Kurtosis Algorithm For Adaptive Filtering of 3D and 4D Signal Processes

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Abstract—In this paper, a novel quaternion adaptive filtering algorithm is proposed for a unified processing of 3D and 4D data, called quaternion least mean kurtosis (QLMK) algorithm. Multidimensional signals exhibit a complex nonlinear relationship and couple among different components. Considering that quaternion has huge advantage in terms of the representation of 3D and 4D signal, quaternion algebra is employed to derive the quaternion least mean square (QLMS) algorithm for hypercomplex signal processes. However, QLMS originates from the least mean square (LMS) algorithm, which may result in performance degradation when the signal is non-Gaussian. Due to the desirable performance of the least mean kurtosis (LMK) algorithm in non-Gaussian situation, in the present work we extend the original LMK algorithm to quaternion domain to manage the 3D and 4D signal processes. The analysis shows that QLMK provides a solution that is responsive to dynamically changing environments. Simulations on prediction of 4D Saito’s chaotic circuit and 3D Lorenz attractor confirm the desirable performance of the proposed method.

Index terms—Adaptive filtering, quaternion, prediction, signal processing.

I. INTRODUCTION

With the developments in adaptive filter theory, many adaptive filtering algorithms are applied to signal processing domain in the past few years [1] [2]. The most well-known algorithms based on mean-square error (MSE) used in practice are the least mean square (LMS) algorithm and its variants [3]-[9].

The LMS is very popular due to its simplicity and efficiency, and the theoretic analytical model can be easily obtained for predicting its convergence behavior. These attractive properties have led to its applications in echo cancelation, radar/sonar signal processing and low delay speech coding [10]. However, the performance of LMS is not always desirable especially for non-Gaussian situations. On the other hand, adaptive filtering algorithms based on non-MSE costs may outperform LMS in some important applications. A typical example is the least mean kurtosis (LMK) algorithm, which was proposed by Tanrikulu and Constantinides[11]. In order to achieve good robustness to noise disturbance, the LMK uses the negative kurtosis of the error signal as the cost function to be minimized. The LMK had been proved to be robust to impulsive, uniformly distributed and nonlinear signal. However, when dealing with multidimensional signals, it processes each dimension of signal separately, which may result in performance loss due to negligence of the correlation among different components. Fortunately, such problem can be solved efficiently by using quaternion to represent 3D or 4D signal.

It has been more than 150 years since the concept of quaternion was first proposed, and the quaternion algebra has been applied in many fields, such as color images [12], integrated inertial navigation and GPS [13], texture segmentation [14], robotics [15] and hypercomplex signal processing [16]. Recently, some adaptive filtering algorithms also have been extended to quaternion domain. Took combines quaternion algebra with LMS to derive the quaternion least square (QLMS) algorithm [17] and the quaternion widely linear (QWL) algorithm [24] and shows their advancements on multidimensional signal processing. Paul introduces [25] the kernel function to QLMS to manage nonlinear quaternion data. In [26] Jahanchahi proposes a diffusion version, called diffusion widely linear quaternion least mean square (D-WLQLMS) algorithm, and demonstrates its robustness to link and node failures on sensor networks. However, due to the squared error cost function, LMS based algorithms may suffer from the problem of performance degradation while the data is non-Gaussian. It’s highly demanded to develop a novel signal processing algorithm for both multi-dimensional and non-Gaussian situation.

In this paper, we enhance the original LMK algorithm by introducing the quaternion algebra. A novel adaptive filtering algorithm, called the quaternion least mean kurtosis (QLMK) algorithm is presented to handle such predicament. QLMK utilizes the negative kurtosis of the error as the cost function to adapt to the more common non-Gaussian data.

The paper is organized as follows. Section 2 briefly reviews the basical principle of quaternion algebra and gives an outline of the LMK algorithm. In Section 3, QLMK algorithm is developed. Section 4 shows simulation performance of the proposed algorithm. Section 5 concludes the paper.

II. BACKGROUND

In this section some related preliminary knowledge will be presented. Some basis of quaternion algebra will be recalled first and then algorithm related to the proposed method will be briefly described.
A. Quaternion algebra

A quaternion can be treated as an extension of a complex number, and the difference is that it consists of four variables, with one scalar part and three imaginary parts [18]. A quaternion variable \( q \) can be expressed as

\[
q = [q, q_i, q_j, q_k] = [a + b i + c j + d k]
\]

where \( a, b, c, d \) are real numbers, and \( i, j, k \) denote the three imaginary parts, adopting the Hamilton rules

\[
ij = k; \quad jk = i; \quad ki = j
\]

\[
i^2 = j^2 = k^2 = -1
\]

From the rules above we can see the noncommutativity of the products of quaternion though they are associated with each other. Other rules of quaternion algebra used in this paper are

\[
q_1q_2 = [a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2, a_1b_2 + a_2b_1 + c_1d_2 - d_1c_2, a_1c_2 + a_2d_1 + b_1d_2 - b_2c_1, a_1d_2 + a_2c_1 + b_2d_1 - b_1c_2]
\]

where \( q = a + bi + cj + dk = [a, b, c, d] \). The symbols “•” and “×” denote, respectively, the dot-product and the cross-product. The conjugation of quaternion \( q \) is

\[
q^* = [a, -b, -c, -d] = a - bi - cj - dk
\]

The quaternion norm is defined by

\[
|q| = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2}
\]

In addition, a quaternion \( q \) is said to be pure imaginary if its real part vanishes. In this case, we have

\[
q = q_i + q_j + q_k
\]

B. LMK algorithm

Consider a linear model

\[
y(n) = w^T(n)x(n)
\]

where \( x(n) \in \mathbb{R}^M \) is an \( M \)-dimensional input vector at instant \( n \), \( w(n) \in \mathbb{R}^M \) is a weight vector with the same dimensionality, \( y(n) \in \mathbb{R} \) is the model output, and \( T \) denotes the transposition operator. Given a sequence of input-desired data \( \{x(n), d(n), n = 1, 2, \ldots\} \), the goal of an adaptive filtering algorithm is to estimate an optimal weight vector \( w \) by updating the weight vector \( w(n) \), such that the difference between model output and the desired signal is as small as possible. Let \( e(n) \) be the error between \( d(n) \) and \( y(n) \)

\[
e(n) = d(n) - w^T(n)x(n)
\]

The cost function of LMK is [11]

\[
J_{\text{LMK}} = 3E\{e^2(n)\} - E\{e^3(n)\}
\]

Based on the cost function, the update equation of the weight vector can be easily derived

\[
w(n + 1) = w(n) - \mu \frac{\partial J_{\text{LMK}}}{\partial w} = w(n) + \mu 4\left[3\sigma^2 - e^2(n)\right]e(n)x(n)
\]

where \( \mu \) denotes the step size, and \( \sigma^2 = E\{e^2(n)\} \) is the mean value of the error square. In most practical situations, \( \sigma^2 \) is usually un-known. However, one can estimate it in an iterative manner

\[
\sigma^2(n) = \beta \sigma^2(n-1) + e^2(n), 0 < \beta < 1
\]

with \( \beta \) being a forgetting factor [19].

III. QLMK ALGORITHM

The proposed QLMK algorithm takes the same cost function (in the quaternion domain) as in the original LMK, that is

\[
J_{\text{QLMK}}(n) = 3E\{|e(n)|^3\} - E\{|e(n)|^4\}
\]

in which

\[
\frac{\partial J_{\text{QLMK}}}{\partial w} = 6E\{|e(n)|^3\} \frac{\partial E\{|e(n)|^4\}}{\partial w} - 4E\{|e(n)|^4\} \frac{\partial E\{|e(n)|^4\}}{\partial w}
\]

Let \( w = w_a + w_i + w_j + w_k \) and \( x = x_a + x_i + x_j + x_k \). We can get

\[
\nabla_v |e(n)| = \nabla_{w_a} |e(n)| + \nabla_{w_i} |e(n)| i + \nabla_{w_j} |e(n)| j + \nabla_{w_k} |e(n)| k
\]

Subsequently, according to the matrix multiplication of quaternion

\[
w^T(n)x(n) = \begin{bmatrix}
    w_x^2 & -w_x w_y & -w_x w_z & -w_x w_t \\
    -w_y w_x & w_y^2 & -w_y w_z & -w_y w_t \\
    -w_z w_x & -w_z w_y & w_z^2 & -w_z w_t \\
    -w_t w_x & -w_t w_y & -w_t w_z & w_t^2 \\
\end{bmatrix}
\]

the derivatives in (16) can be computed as

\[
\nabla_v |e(n)| = -\nabla_{w_a} |e(n)| + \nabla_{w_i} |e(n)| i + \nabla_{w_j} |e(n)| j + \nabla_{w_k} |e(n)| k
\]

Substituting (18) - (21) into (16), we get

\[
\nabla_v |e(n)| = \nabla_{w_a} |e(n)| i + \nabla_{w_b} |e(n)| j + \nabla_{w_c} |e(n)| k = 2x^2(n)
\]

Thus, the iterative equation for updating the weight vector is

\[
w(n + 1) = w(n) - \mu \frac{\partial J_{\text{QLMK}}}{\partial w} = w(n) + 8\mu \left( |e(n)| - 3E\{|e(n)|^4\} \right) |e(n)| x^T(n)
\]
where \( E\{|e(n)|^2\} = \beta E\{|e(n-1)|^2\} + |e(n)|^2, 0 < \beta < 1 \). If the imaginary parts \( i \) and \( j \) of the quaternion signal \( x(n) \) vanish, the above update equation will become
\[
w(n+1) = w(n) + 8\mu \left[ 3E\{|e(n)|^2\} - |e(n)|^2 \right] e(n)(x_j + x_k)
\] (24)

It shows that QLMK does not simplify exactly into the complex-domain LMK. However, when the signal is three-dimensional, the weight update equation will conform with the equation (23).

The step size \( \mu \) is a significant parameter in QLMK as it governs the convergence behaviors. Although one can theoretically derive a range of step size for guaranteeing the convergence, it is still difficult to determine an appropriate step size, as it requires eigendecomposition of the correlation matrix [20]. However, the left eigenvalue decomposition of a quaternion is an ongoing research topic [21]. In practical applications, the step size can be manually set or chosen by trial and error methods.

IV. SIMULATIONS

In order to demonstrate the performance of the proposed QLMK algorithm, experiments are given in the following to forecast two kinds of non-Gaussian signal, 4D Saito's chaotic circuit and 3D Lorenz attractor. Our simulations were conducted for M-step ahead prediction, which means using the current input to predict the state of the input signal after M steps. Components of the signal were combined into one quaternion valued signal in the simulation.

For rigor, the performance of QLMK was compared with multiple LMK and QLMS. Similar to the multiple LMS algorithm [22], the \( j \)^{th} output of the four-channel LMK adaptive filter is \( y_j(n) = w_j(n)x_j(n), j = 1, ..., 4 \), where \( w_j(n), x_j(n), y_j(n) \) donate respectively, the weight vector, input vector and output of the \( j \)^{th} channel. The update equation for each weight vector is
\[
w_j(n+1) = w_j(n) + \mu \left[ 3\sigma_j^2(n) - e_j^2(n) \right] e_j(n)x_j(n), j = 1, ..., 4\) (25)
where \( \sigma_j^2(n) = \beta \sigma_j^2(n-1) + e_j^2(n), 0 < \beta < 1 \).

A. Simulation Case 1: Prediction of 4D Saito's Chaotic Circuit Signal

The Saito's chaotic circuit is governed by four state variables and five parameters, and is given by [23]
\[
\begin{bmatrix}
\frac{\partial X_1}{\partial \tau} \\
\frac{\partial X_2}{\partial \tau} \\
\frac{\partial X_3}{\partial \tau} \\
\frac{\partial X_4}{\partial \tau}
\end{bmatrix} = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
-\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\
-\alpha_5 & -\alpha_6 & -\alpha_7 & -\alpha_8 \\
-\alpha_9 & -\alpha_{10} & -\alpha_{11} & -\alpha_{12}
\end{bmatrix} \begin{bmatrix}
X_1 - \eta \rho \eta h(z) \\
X_2 - \eta \rho \eta h(z) \\
X_3 - \eta \rho \eta h(z) \\
X_4 - \eta \rho \eta h(z)
\end{bmatrix} - \begin{bmatrix}
Y_1 - \eta \rho \eta h(z) \\
Y_2 - \eta \rho \eta h(z) \\
Y_3 - \eta \rho \eta h(z) \\
Y_4 - \eta \rho \eta h(z)
\end{bmatrix}
\] (26)

where \( \tau \) is the time constant of the chaotic circuit and \( h(z) \) is the normalized hysteresis value which is

\[
h(z) = \begin{cases} 
1, z \geq -1 \\
-1, z \leq 1 
\end{cases}
\] (27)

Also, the variables \( z, \rho_1 \) and \( \rho_2 \) are
\[
z = X_1 + X_2, \rho_1 = \frac{\beta_1}{1 - \beta_1}, \rho_2 = \frac{\beta_2}{1 - \beta_2}
\] (28)

Fig. 1 shows the 4D Saito's chaotic signal with the standard parameters setting: \( \eta = 1.3, \alpha_1 = 7.5, \alpha_2 = 15, \beta_1 = 0.16, \beta_2 = 0.097 \).
The corresponding convergence curves for the one step ahead prediction when the filter length is 5 are plotted in Fig.2. In the simulation, the step sizes are experimentally set at $\mu = 5 \times 10^{-6}$ and for LMK and QLMK, the forgetting factor $\beta$ is 0.01. Fig.3 demonstrates the steady-state MSEs (calculated using the sum of 200 samples at final steady stage) of LMK, QLMS and QLMK for different filter lengths ($L$) and prediction horizons ($M$) with $\mu = 5 \times 10^{-6}, \beta = 0.01$. Evidently, QLMK has the potential to simultaneously beat LMK and QLMS in terms of the convergence rate and the steady-state prediction MSE.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Training MSE</td>
<td>0.0083</td>
<td>0.002</td>
<td>0.0013</td>
<td>0.0007</td>
<td>0.0003</td>
<td>0.0005</td>
<td>0.0007</td>
</tr>
<tr>
<td>Testing MSE</td>
<td>0.0152</td>
<td>0.0051</td>
<td>0.0034</td>
<td>0.0012</td>
<td>0.001</td>
<td>0.0026</td>
<td>0.0047</td>
</tr>
</tbody>
</table>

In addition, to show the effect of the forgetting factor on the performance of prediction, we set $\beta$ to different values. The training and testing errors measured at the steady state (calculated using the average of 200 samples at final steady stage) are shown in Table I (with $M = 1, L = 5, \mu = 5 \times 10^{-6}$). The weight vector updates with iterations in the training process, and in the testing process the weight vector keeps the optimal weight vector $w_{opt}$ obtained from the training process. As one can see, with a proper forgetting factor ($\beta = 0.3$ in this example), the proposed method can achieve satisfying performance.

B. Simulation Case 2: Prediction of 3D Lorenz attractor

As a typical three dimensional nonlinear and non-Gaussian signal, Lorenz attractor has been used to model lasers, dynamos, and the motion of waterwheel. The following equations govern the Lorenz attractor system

$$\frac{dX}{dt} = \alpha(Y - X), \quad \frac{dY}{dt} = X(\rho - Z) - Y, \quad \frac{dZ}{dt} = XY - \beta Z$$  (29)

where $\alpha > 0, \rho > 0, \beta > 0$ and $t$ is the time constant of the system. In this example, the three parameters are $\alpha = 10, \rho = 28, \beta = 8/3$ and 3D Lorenz attractor signal is shown in Fig.4. Further, for different filter lengths ($L$) and prediction horizons ($M$), the steady-state MSEs (calculated using the sum of 200 samples at final steady stage) are illustrated in Fig.5. The step sizes are $\mu_{LMK} = 5 \times 10^{-11}, \mu_{QLM} = 1 \times 10^{-8}$ and $\mu_{QLMK} = 2 \times 10^{-11}$ in this simulation. For LMK and QLMK, the forgetting factor $\beta$ is 0.7. It can be seen that the proposed method outperforms the LMK method and QLMS method as it achieves the smallest prediction MSE.

V. CONCLUSION

In this work we present a novel adaptive filtering algorithm, named QLMK, for 3D and 4D signal processes. We extend the least mean kurtosis (LMK) algorithm to the quaternion domain to overcome the performance degradation of the existing methods in non-Gaussian multidimensional signal processes. Simulations on prediction for 4D Saito's chaotic circuit and 3D Lorenz attractor are presented. Our study shows that the proposed method has obvious advantages in both convergence rate and steady-state MSE over the original quaternion least mean square (QLMS) and four-channel LMK algorithm.

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