Microannular electro-osmotic flow with the axisymmetric lattice Boltzmann method
G. H. Tang, X. F. Li, and W. Q. Tao

Citation: Journal of Applied Physics 108, 114903 (2010); doi: 10.1063/1.3517437
View online: http://dx.doi.org/10.1063/1.3517437
View Table of Contents: http://scitation.aip.org/content/aip/journal/jap/108/11?ver=pdfcov
Published by the AIP Publishing

Articles you may be interested in
Effects of non-Newtonian power law rheology on mass transport of a neutral solute for electro-osmotic flow in a porous microtube
Biomicrofluidics 7, 044113 (2013); 10.1063/1.4817770

Alternating current electroosmotic flow of the Jeffreys fluids through a slit microchannel

Electro-osmotic flow in a wavy microchannel: Coherence between the electric potential and the wall shape function
Phys. Fluids 22, 082001 (2010); 10.1063/1.3467035

Time periodic electro-osmotic flow through a microannulus
Phys. Fluids 22, 042001 (2010); 10.1063/1.3358473

Electroosmotic flow and mixing in microchannels with the lattice Boltzmann method
J. Appl. Phys. 100, 094908 (2006); 10.1063/1.2369636
Microannular electro-osmotic flow with the axisymmetric lattice Boltzmann method

G. H. Tang, a X. F. Li, and W. Q. Tao
State Key Laboratory of Multiphase Flow, School of Energy and Power Engineering, Xi’an Jiaotong University, Xi’an 710049, China

(Received 20 August 2010; accepted 21 October 2010; published online 7 December 2010)

Considering electro-osmotic flow in axisymmetric microducts is of both fundamental interest and practical significance. In this paper, an axisymmetric lattice Boltzmann model which solves the complete nonlinear Poisson–Boltzmann equation is presented to obtain the electric potential distribution in the electrolytes, and another axisymmetric lattice Boltzmann model is employed to solve the velocity field. First, the lattice Boltzmann model is validated by the electric potential distribution in the electrolyte with analytical solutions and finite volume method. Second, velocity distributions in circular tubes at various conditions are discussed. Then, we extend the lattice Boltzmann model for steady and pulsating electro-osmotic flow through annular microducts and the influences of inner to outer radius ratio, inner to outer zeta potential ratio and oscillating frequency are investigated. In addition, a numerical study of electro-osmotic flow in circular and annular microducts considering the non-Newtonian fluid behavior is also conducted for the first time. The results show that the microannular electro-osmotic flow exhibits much difference from the planar flow and the non-Newtonian rheology has significant effect on the flow behavior as well. © 2010 American Institute of Physics. [doi:10.1063/1.3517437]

I. INTRODUCTION

Recently, the microfabrication technology has led to a rapid development of microelectromechanical systems (MEMS) and laboratory-on-a-Chip (LOC) devices. 1 The technological demands on microfluidic systems require a better understanding of the microscale fluidic phenomena. For microscale liquid flow, electrokinetic effect is one of the most important surface effects. In this study we focus on one of the electrokinetic phenomena, electroosmosis. Electroosmotic flow (EOF) is generated due to the interactions of ions in the electrical double layer (EDL) with an externally applied electrical field. 2 Electroosmosis is a very promising nonmechanical technology for pumping, separating, and mixing in microdevices. 3 Due to its wide applications, EOF in microchannels has attracted much attention. 1–10

The recent development of the lattice Boltzmann method has provided an alternative simulation tool for computational fluid dynamics. 11 Due to its distinctive advantages over conventional numerical methods, the lattice Boltzmann method has been successfully applied to different flows in science and engineering as an efficient and promising numerical method. Recently, the lattice Boltzmann method has been also found success in applications to microfluidics. Particularly, the lattice Boltzmann method has been successfully applied in modeling a range of microfluidic electro-osmotic applications. 12–16 However, some of studies employed a linearized Poisson–Boltzmann (PB) equation or solved the nonlinear PB equation with hybrid methods. These limitations are noted by many researchers, 3,6–8,10 and improved lattice Boltzmann models are proposed in succession. However, the existing lattice Boltzmann models for EOF are limited to parallel plate channels or parallel slits. Therefore, it is desirable to propose an axisymmetric lattice Boltzmann method for EOF in microducts. In addition, the flow behavior of non-Newtonian fluid in microannular ducts is of high interest in many areas of science and technology. 17 Fundamental understanding of the EOF of non-Newtonian fluid in microcircular and annular ducts is important to correctly predict the performance of microfluidic devices. In our previous work, 18 we studied EOF of non-Newtonian fluid in parallel microchannels. In this article, we extend the work to EOF of non-Newtonian fluid in circular and annular ducts using the present axisymmetric lattice Boltzmann method.

In recent years, the axisymmetric lattice Boltzmann method (LB) for axisymmetric flows has attracted much attention, 19–27 in which we found that Zhou’s model 24 is fairly simple and efficient. Huang and Lu 28 recently also shown that Zhou’s model seems to be the simplest one and this model is most stable in terms of τ sensitivity through theoretical and numerical analyses of three types of axisymmetric lattice Boltzmann models. In this regard, we employed Zhou’s model to solve the velocity field. Note that some limitations still exist in the model such as neglecting azimuthal velocity, and containing some source terms involved with velocity gradients. Following the spirit of axisymmetric thermal lattice Boltzmann method in our previous work, 27 we develop an axisymmetric lattice Boltzmann model for the complete nonlinear PB equation to obtain the electric potential distribution in the electrolyte.

The rest of the paper is organized as follows. In Sec. II, the macroscopic governing equations for EOF are introduced. In Sec. III, we present the discrete Boltzmann equation for velocity field and electric potential. Section IV pre-
resents numerical results of electric potential and velocity distribution for EOF in circular tubes and annular ducts. A brief conclusion is presented at the end.

II. MACROSCOPIC HYDRODYNAMIC EQUATIONS FOR EOF

Consider an incompressible Newtonian aqueous electrolyte with ions continuously distributed in the whole microduct of radius $R$. The electric potential $\psi$ at a point distance $r$ from the axis can be described by the following Poisson equation in the cylindrical coordinate system:

$$\nabla^2 \psi(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = -\frac{\rho_e(r)}{\varepsilon \varepsilon_0},$$  \hspace{1cm} (1)

where $\rho_e(r)$ is the net charge density at a point distance $r$ from the axis, $\varepsilon$ is the relative dielectric constant of the solution, and $\varepsilon_0$ is the permittivity of a vacuum. The ion distribution can be expressed by the Boltzmann equation

$$n_i = n_{i,\infty} \exp(-z_i e \psi/k_B T),$$  \hspace{1cm} (2)

where $n_{i,\infty}$ is the ionic number concentration of $i$th species, $z_i$ is the valence of type $i$ ions, and $n_\infty$ is the ionic number concentration in the bulk solution. The bulk ionic concentration $n_\infty$ can be expressed as the product of the ionic molar concentration of $c_\infty$ with the Avogadro’s number of $N_0$, i.e., $n_\infty = c_\infty N_0$. Constant $e$ is the charge of a proton, $k_B$ is the Boltzmann constant, and $T$ is the absolute temperature. For a symmetric electrolyte ($z_i = z$ and $n_{i,\infty} = n_\infty$) considered in this study, the net charge density can be expressed as

$$\rho_e(r) = -2n_\infty z e \sinh\left(\frac{z_i e \psi}{k_B T}\right).$$  \hspace{1cm} (3)

Substituting Eq. (3) into Eq. (1) yields the nonlinear PB equation

$$\nabla^2 \psi(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = \frac{2n_\infty z e}{\varepsilon \varepsilon_0} \sinh\left(\frac{z_i e \psi}{k_B T}\right).$$  \hspace{1cm} (4)

If the term $\frac{z_i e \psi}{k_B T}$ is small, we have $\sinh\left(\frac{z_i e \psi}{k_B T}\right) \approx z_i e \psi/k_B T$, which is known as Debye–Hückel approximation. Then Eq. (4) recovers the following linearized equation:

$$\nabla^2 \psi(r) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = \kappa^2 \psi,$$  \hspace{1cm} (5)

where $\kappa = (2n_\infty z^2 e^2/\varepsilon \varepsilon_0 k_B T)^{1/2}$, the reciprocal of Debye length. The boundary conditions on $\psi(r)$ are, as follows:

$$\frac{d\psi(r)}{dr} \bigg|_{r=0} = 0 \quad \text{and} \quad \psi(r) \bigg|_{r=R} = \zeta,$$  \hspace{1cm} (6)

where $\zeta$ is the zeta potential at the wall. Therefore, the solution of the linearized PB equation is

$$\psi(r) = \frac{I_0(\kappa R)}{I_0(\kappa R)} \zeta,$$  \hspace{1cm} (7)

where $I_0$ is the zero-order modified Bessel function of the first type. Equation (3) then becomes

$$\rho_e(r) = -2n_\infty z e \sinh\left(\frac{z_i e \psi}{k_B T}\right) = -2n_\infty z e \sinh\left(\frac{z_i e \psi_0}{k_B T}\right),$$  \hspace{1cm} (8)

When the cylindrical tube is applied with an external electric field along the axis direction, the motion of aqueous electrolyte is governed by the Navier–Stokes equations, which can be reduced to the following equation if assuming that the pure electrical driven flow is fully developed:

$$\frac{\partial u}{\partial t} = -\eta \frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) + \rho_e(r) E_z,$$  \hspace{1cm} (9)

where $u$ is the streamwise velocity, $\eta$ is the dynamic viscosity, $\rho_e$ is the density of the fluid, $\rho_e$ is the local net charge density, and $E_z$ is the external electric field intensity applied along the axis direction. Considering the nonslip velocity at the wall and the geometry symmetry, the boundary conditions on $u(r)$ are, as follows:

$$\frac{du(r)}{dr} \bigg|_{r=0} = 0 \quad \text{and} \quad u(r) \bigg|_{r=R} = 0.$$  \hspace{1cm} (10)

The velocity $u(r)$ can be solved by combining Eq. (9) with Eqs. (8) and (10) for a steady-state flow:

$$u(r) = -\frac{\varepsilon \varepsilon_0 z}{\eta} E_z \left[1 - \frac{I_0(\kappa r)}{I_0(\kappa R)}\right].$$  \hspace{1cm} (11)

III. NUMERICAL METHODS

A. The lattice Boltzmann equation for velocity field

In this study, the Zhou’s axisymmetric lattice Boltzmann method is employed to solve the velocity field, the discrete evolution equation is

$$f_a(x + c_a \delta x, t + \delta t) = f_a(x, t) - \frac{1}{\tau_v} (f_a - f_{eq}^a) + \frac{\delta}{\gamma c_s c_{ax}} F_x,$$  \hspace{1cm} (12)

where $f$ is the single particle density distribution function and $f_{eq}$ is the corresponding equilibrium distribution function, $\tau_v$ is the nondimensional relaxation time, $c_a$ is the particle discrete velocity. In this model, we have $c_0 = 0$, $c_a = \cos(\alpha - 1/2/\pi/2, \sin(\alpha - 1/2/\pi/2)|c$ for $\alpha = 1, 2, 3, 4$ and $c_a = \cos(\alpha - 5/2/\pi/2 + \pi/4, \sin(\alpha - 5/2/\pi/2 + \pi/4)|\sqrt{2} c$ for $\alpha = 5, 6, 7, 8$, where $c = \delta_{\parallel} / \delta_{\perp}$ is the particle streaming speed ($\delta_{\parallel}$ and $\delta_{\perp}$ are the lattice spacing and time step, respectively). $c_{ax}$ is the particle’s velocity in $x$ link. $\gamma$ is a constant, here $\gamma = 6$, $F_x$ is the force term in $x$ direction which is given by Eq. (13),

$$F_x = \frac{\rho \varepsilon_0 \partial u}{r \partial r} + E_z \rho_e(r).$$  \hspace{1cm} (13)

The equilibrium density distribution function is defined as

$$f_{eq}^a = \rho \omega_a \left[1 + \frac{3}{c_s^2} (c_a \cdot u) + \frac{9}{2c_s^4} (c_a \cdot u)^2 - \frac{3}{2c_s^2} (u \cdot u)\right],$$  \hspace{1cm} (14)

where $\omega_0 = 4/9$, $\omega_a = 1/9$ for $\alpha = 1, 2, 3, 4$ and $\omega_a = 1/36$ for $\alpha = 5, 6, 7, 8$. The macroscopic variables such as mass density and momentum density can be obtained by summing over the distribution function $f_a(x, t)$, as follows:
\[ \rho = \sum_a f_a \quad \text{and} \quad \mathbf{u} = \sum_a f_a \mathbf{c}_a / \rho. \]  

(15)

The kinematic viscosity is

\[ \nu = \frac{c^2 \delta_i}{3}(\tau_v - 0.5). \]  

(16)

B. The lattice Boltzmann equation for electric potential

Comparing the nonlinear PB equation [Eq. (4)] with the special case of the steady-state laminar incompressible energy equation when the velocity is zero and the thermal diffusivity is unity, we can derive the following discrete lattice Boltzmann evolution equation for the electric potential similar to the treatment of the energy equation in Ref. 27. One can refer to the appendix for details, as follows:

\[ g_a(\mathbf{x} + \mathbf{c}_a \delta_t, t + \delta_t) - g_a(\mathbf{x}, t) = - \omega_g[g_a(\mathbf{x}, t) - g^{eq}_a(\mathbf{x}, t)] \]

\[ + (1 - 0.5 \omega_g) \delta_S \psi(\mathbf{x}, t), \]

(17)

where \( g \) is the distribution function and \( g^{eq} \) is the corresponding equilibrium distribution function, with \( g^{eq}_0 = 4 \psi / 9, g^{eq}_{1,2,3,4} = 0, g^{eq}_{5,6,7,8} = (1/36). \) The potential diffusivity \( \chi \) which is equal to unity in the simulations, is defined as

\[ \chi = \frac{\delta_S \tau_c e^2}{3}. \]

(18)

The source term is chosen as

\[ S_1 = - \frac{2 n_s e}{9 \varepsilon \varepsilon_0} \sinh \left( \frac{z e \psi}{k_B T} \right). \]

(19)

\( \omega_g \) is given by

\[ \omega_g = \left[ 1 + (c_{\alpha} \tau_g \delta_t / r) \right] / (\tau_g + 0.5). \]

(20)

The macroscopic electric potential in the liquid is calculated from Eq. (21) with an iteration method,

\[ \psi = \sum_a g_a - 0.5 \delta_t \frac{2 n_s e z e}{\varepsilon \varepsilon_0} \sinh \left( \frac{z e \psi}{k_B T} \right). \]

(21)

At the same time, we also derive the discrete lattice Boltzmann evolution equation for solving the linear PB equation [Eq. (5)], as follows:

\[ g_a(\mathbf{x} + \mathbf{c}_a \delta_t, t + \delta_t) - g_a(\mathbf{x}, t) = - \omega_g[g_a(\mathbf{x}, t) - g^{eq}_a(\mathbf{x}, t)] \]

\[ + (1 - 0.5 \omega_g) \delta_S \psi(\mathbf{x}, t). \]

(22)

The corresponding source term is chosen as

\[ S_2 = - \kappa^2 g^{eq}_a. \]

(23)

Note that though there are some other sophisticated methods to solve the PB equation with high efficiency, such as conventional finite volume or finite difference numerical method, the convergence of the present lattice Boltzmann method is acceptable and the iterative computation reaches a local equilibrium within a short time. The present approach aids to solve all the equations within the same LBM framework, which is especially efficient for multiprocessor computing. Similar idea also appeared in Ref. 3 for parallel channel simulation. In addition, in this model, we use a uniform dielectric constant and neglect the fluctuations in the dielectric constant and thus the net charge density distribution in a symmetric electrolyte is proportional to the concentration difference between the cations and anions. This assumption is limited to small Peclet number less than one hundred when the ion-convection effect is insignificant compared to ion diffusion.

C. The boundary conditions for the lattice Boltzmann equations

Attention is now turned to the boundary conditions. Due to its kinetic nature and essential character of microscopic physics, the lattice Boltzmann method is of good capability for handling more general boundary conditions. Here we adopted the nonslip boundary conditions by Zou and He31 on the upper and bottom walls to solve the density distribution function for velocity field. With regard to the lattice Boltzmann equation for the electric potential, the zeta potential on the upper and bottom walls are fixed in the simulation and Dirichlet boundary conditions are employed.32 For the microcircular tube inlet and outlet, we employed periodic boundary conditions by assuming a fully developed flow.

IV. RESULTS AND DISCUSSION

A. The electro-osmotic potential distribution in the electrolytes

The electrolyte confined in a microcircular tube of radius \( R=2 \ \mu m \) is considered. Unless otherwise mentioned in this paper, \( \varepsilon \varepsilon_0 = 7.79 \times 10^{-10} \ \text{C}^2/(\text{J} \text{ m}) \), \( T=293 \ \text{K}, e = 1.6 \times 10^{-19} \ \text{C}, N_A = 6.02 \times 10^{23} \ \text{mol}^{-1}, k_B = 1.38 \times 10^{-23} \ \text{J/K}, \rho = 998.2 \ \text{kg/m}^3, z=+1, \nu = 1.004 \times 10^{-6} \ \text{m}^2/\text{s}. \) Figure 1 shows the electric potential distribution across the pipe. In this case, the bulk ionic concentration is \( c_s = 10^{-5} \ \text{M}. \) The solid symbols in the figure represent the analytical solutions of the linearized PB equation, the dashed lines represent LB results for the linearized PB equation, the solid lines represent the LB results for the nonlinear PB equation and the open symbols presents the results by numerically solving the nonlinear PB equation with our own developed code of the finite volume method (FVM). The electric potential in the electrolytes across the circular tube is decreasing from a given wall zeta potential (commonly assumed to be equal to the electrostatic surface potential), to zero around center zone. We can see that all the four results are in good agreement when the zeta potential is small, say, less than 50 mV in...
the discrepancy between the nonlinear and linearized results increases as the zeta potential increases, though the LB results of the linearized equation still agree well with the linearized analytical solution, which confirms that the linearization of the PB equation is limited to small zeta potential. Our LB solutions for nonlinear PB equation are in good agreement with the macroscale FVM numerical results for all the cases.

**B. EOF in circular tubes**

We consider a steady flow driven by a uniform external electric field in a circular tube. The effects due to the variations of the ionic concentration, tube radius and external electric field are investigated with the present axisymmetric lattice Boltzmann method. A uniform external electric field $E_x$ is applied along the axis direction of the tube $R=2$ μm. The zeta potential at the radius surface keeps at ζ = −50 mV and the external electric field intensity keeps at $E_x=20 000$ V/m. Figure 2 shows the streamwise velocity across the circular tube for the various ionic concentrations. From the upper to the bottom, the bulk ionic molar concentration is $c_ω=10^{-3}$, $10^{-4}$, $10^{-5}$, $10^{-6}$, $10^{-7}$, and $10^{-8}$ M and the corresponding Debye length is 0.01, 0.032, 0.1, 0.32, 1, and 3.2 μm. Namely, as the bulk ionic concentration decreases, the Debye length increases. The velocity profiles across the tube resemble typical parabolas as the magnitude of the Debye length increases. From the figure we can also see that for the shown bulk ionic concentration range of $c_ω=10^{-3}–10^{-6}$ M, the Debye length is small and the EDL is restricted to a very narrow region close to the wall. Therefore, the velocity increases sharply very near the wall and then almost keeps a constant in most of the tube region. More specifically, the ratio of the tube diameter to the Debye length is larger than about 10 for this bulk ionic concentration range of $c_ω=10^{-3}–10^{-6}$ M.

The effect of external electric field on EOF is studied under the condition of the ionic concentration $c_ω=10^{-6}$ M, zeta potential at radius surface ζ = −50 mV and tube radius $R=2$ μm. The effect of external electric field on velocity profile is shown in Fig. 3. It is found that as the external electric field intensity increases, the EOF velocity almost increases proportionally.

Figure 4 shows the electro-osmotic velocity for various radii under the condition of the bulk ionic concentration $c_ω=10^{-6}$ M, zeta potential $\zeta =-50$ mV and external electric field intensity $E_x=20 000$ V/m. From Fig. 4 we find that the velocity distributions in the tube center region and the maximum velocity nearly keep a constant if the ratio of the tube diameter to the Debye length is larger than about 10 ($R/\lambda$).
C. Steady EOF in annular ducts

The axisymmetric lattice Boltzmann method used above is also extended to simulate the EOF in annular ducts with the inner radius $R_1$ and outer radius $R_2$. The boundary conditions are that the inner circular wall is subjected to an electric potential $\psi_1$ and the outer circular wall with an electric potential $\psi_2$.

Consider EOF in an annular duct for $\psi_1 = \psi_2 = -50$ mV, $c_\infty = 10^{-7}$ M, $R_2/R_1 = 0.4$, and $E_x = 20,000$ V/m. The distributions of electrical potential in the annulus are very important to the understanding of the flux in the annulus. Figure 5 shows the electric potential variation along the radial coordinate with different values of $\kappa R_2$. Note that we use the dimensionless parameters, the inner to outer radius ratio $R_1/R_2$, the inner to outer zeta potential ratio $\psi_1/\psi_2$, and the ratio of radius to Debye length $\kappa R_2$ due to the impact of multiparameter in annular ducts. It is found that the electric potential profiles across the annulus develop sharp features in the near vicinity of the wall as $\kappa R_2$ increases. The electric potential gets smaller as $\kappa R_2$ decreases. This phenomenon is also caused by the overlapping of Debye layers from the inner surface and the outer surface when the annulus gap becomes smaller. Figure 6 shows the corresponding velocity profiles for various $\kappa R_2$. From the figure we can see that as $\kappa R_2$ increases, the velocity profiles across the tube also develop sharp features in the near vicinity of the wall and then keep a constant nearly in most of the annulus region when the value of $\kappa R_2$ is large ($\kappa R_2 = 20$ in the figure). While the velocity profiles across the annulus present to be typical parabolas when the magnitude of $\kappa R_2$ is small.

Figures 7 and 8 show the electric potential distribution in annular ducts when the inner and outer surfaces have different zeta potentials. From these figures, we can see that the electric potential along the radial direction at a certain point becomes smaller as the ratio of inner radius to outer radius decreases. Figure 7 shows the electric potential distribution in annular duct at a large $\kappa R_2 = 10$. We can see that when the radius ratio is small, the magnitude of electric potential decreases first and then increases from the surface with smaller zeta potential to the other surface with larger zeta potential. However, for large radius ratio $[R_1/R_2 = 0.9$ in Fig. 7(a) and
$R_1/R_2=0.8$ in Fig. 7(b), the electric potential always increases from the surface with lower zeta potential to the surface with larger zeta potential. Figure 8 shows the electric potential distribution in annular duct at a small $\kappa R_2=1$. We can see that the magnitude of electric potential always increases from the surface with smaller zeta potential to the other surface with larger zeta potential at the shown radius ratios. Comparing Fig. 7 with Fig. 8, we can see that at large $\kappa R_2$, the differences in the electric potential among different ratios of inner to outer radius are more significant than those at small $\kappa R_2$. The results are in good agreement with those reported in Ref. 33.

Figures 9 and 10 show the corresponding velocity profiles in annular ducts. We find that the streamwise velocity becomes smaller as the ratio of inner radius to outer radius increases. There occurs a maximum velocity for every radius ratio. This maximum is closer to the radius surface with larger zeta potential but approaches the midpoint between cylinders as the gap $R_2-R_1$ becomes small (i.e., the radius ratio of $R_1/R_2$ increases). In addition, as the gap becomes small, the velocity profile approaches a parabolic distribution. Comparing Fig. 10 (small $\kappa R_2=1$) with Fig. 9 (large $\kappa R_2=10$), we can see that at small $\kappa R_2$ the maximum velocity appears closer to the annulus midpoint for the same radius ratio.

### D. Electro-osmotic pulsating flow in annular ducts

We study the EOF flow in annular microducts by changing the polarity of the externally imposed electric field. We used very high frequency electric fields to drive the flow. The time periodic external electric intensity is

$$E(t) = \Re [E_0 e^{i\omega t}],$$

with an amplitude $E_0$ and a frequency $\omega=2\pi/(T_{E}\delta)$. In the simulations, the time step is $\delta t = 2.7 \times 10^{-12}$ s, and $T_{E}$ represents the period of the externally imposed electric field normalized by $\delta$. The bulk ionic concentration is $c_i=10^{-4}$ M and the corresponding Debye length is $0.032$ $\mu$m. The ratio of inner radius $R_1$ to outer radius $R_2$ is 0.4, and the external electric field is $E_x=20 000$ V/m.

In Fig. 11, the electric potential at the inner and outer wall both keeps the same at $\psi_1=\psi_2=-50$ mV. From Figs. 11(a)–11(c), as the applied frequency increases, the perturbed flow region becomes smaller. For the lower frequency the velocity perturbation propagates from the near-wall region deeper into the bulk region while for even higher fre-
quency, the flow in center region remains virtually unperturbed. This phenomenon is attributed to that when the frequency is very high, the electric field changes its direction quite rapidly and the fluid cannot respond fast enough due to its finite inertia. The diffusion time scale is greater than the oscillation time period and hence no sufficient time for the frequency is very high, the electric field changes its direction disturbed. This phenomenon is attributed to that when the frequency, the flow in center region remains virtually unperturbed. The presented EOF pulsating results would be useful in practical applications of controlling mass flow rate or species mixing.

E. EOF of non-Newtonian fluid

The electro-osmotic flows studied above are limited to Newtonian fluid. In this section, we investigate flow of non-Newtonian fluid in circular and annular pipes using the present modified axisymmetric lattice Boltzmann model. In the commonly used power-law model for non-Newtonian fluid, the viscosity is given by

\[ \eta = \eta_0 \gamma^n, \]

where the shear rate related \( \gamma \) is \( \gamma = |du/dr| \) and the parameter \( n \) is the power-law exponent which determines the response of the fluid to change in shear rate. The fluid is shear thinning for \( n < 1 \), shear thickening for \( n > 1 \) and Newtonian fluid with shear-independent viscosity \( \eta_0 \) for \( n = 1 \). Coupling Eqs. (16) and (26), together with \( \eta = \rho \eta_0 \), we have a shear-dependent relaxation time \( \tau_r \) at each node in the lattice Boltzmann evolution Eq. (12).

A simple electro-osmotic driven non-Newtonian fluid flow in microcircular tube of radius \( R = 2 \mu m \) is considered. In the simulation, we choose \( \varepsilon_0 = 7.79 \times 10^{-10} C^2/(Jm) \), \( T = 293 K \), \( e = 1.6 \times 10^{-19} C \), \( N_A = 6.02 \times 10^{23} \text{mol}^{-1} \), \( k_B = 1.38 \times 10^{-23} J/K \), \( \rho = 1000 \text{kg/m}^3 \), \( Z = +1 \), and \( \nu = 1.004 \times 10^{-5} \text{m}^2/s \), the surface zeta potential of \( \zeta = -50 \text{mV} \), and the externally applied electric field intensity of \( E_R = 20 \times 10^3 \text{V/m} \). The treatment of the boundary conditions is the same as the Newtonian fluid flow in circular pipe.

Figures 13(a) and 13(b) present the normalized streamwise velocity profile along the diameter of pipe for \( c_o = 10^{-8} \text{M} \) and \( c_o = 10^{-5} \text{M} \), respectively. From the figure, we can see that for the Newtonian fluid of \( n = 1 \), the velocity profile is parabolic at lower ionic concentration, which resembles a typical pressure-driven flow. However, as the bulk ionic concentration increases, the velocity profile transits from parabolic to pluglike. The velocity profiles for non-Newtonian fluid of \( n = 1.3 \) and \( n = 1.5 \) show curvature and sharplike around the center peak velocity while the case of \( n = 0.6 \) presents typical plug flow for \( c_o = 10^{-8} \text{M} \). The velocity profiles for non-Newtonian fluid of \( n = 1.2 \) and \( n = 1.4 \) show curvature and sharplike around the center peak velocity while the case of \( n = 0.6 \) presents typical plug flow for \( c_o = 10^{-5} \text{M} \). The velocity profiles for all the values of \( n \) exhibit more flattening at \( c_o = 10^{-5} \text{M} \) compared to \( c_o = 10^{-8} \text{M} \). These different flow profiles are caused by variations in the Debye length. The calculated Debye length for \( c_o = 10^{-5} \text{M} \) is \( 0.10113 \mu m \), compared to \( 3.19747 \mu m \) for \( c_o = 10^{-8} \text{M} \). We also studied the electro-osmotic driven non-Newtonian fluid flow in microannular tube of fixed outer radius \( R_2 = 5 \mu m \). The other parameters here are the same as the above case. As shown in Fig. 14, for \( n = 0.6 \) we observe a general flattening of the velocity profile. Conversely, the profiles for \( n = 1.2 \) and \( n = 1.4 \) show great curvature near the center peak velocity, and the larger exponent \( n \), the greater curvature. From Figs. 14(a) and 14(b), the velocity profiles for all the values of \( n \) exhibit more flattening at \( c_o \).
= $10^{-5}$ M compared to $c_0 = 10^{-8}$ M. In Fig. 14(c) we can see that the velocity profile leans from the inner surface to the annulus center as the value of $n$ increases. Comparing Fig. 14(c) with Fig. 14(a), we can see that the velocity profile leans toward the inner cylinder as the radius ratio of inner radius to outer radius decreases.

**V. CONCLUSIONS**

Motivated by the growing interest in electroosmosis as a reliable no moving parts strategy to pump liquid, to mix liquid, and to control fluid motion in microfluidic devices, we have numerically studied steady-state and pulsating EOF in circular and annular microducts. We have developed an axisymmetric lattice Boltzmann model, which recovers the complete nonlinear PB equation, to solve the electric potential distribution in the electrolytes. And we used another axisymmetric lattice Boltzmann model, which recovers the incompressible Navier–Stokes equation, to solve the velocity field. We have validated the method by the electric potential distribution with analytical solutions and FVM numerical results.

We first applied this method for EOF in circular tubes. By studying the effects of the bulk ionic concentration, external electric field and tube radius, we find that the velocity profiles exhibit a sharp variation adjacent to the walls and the maximum velocity nearly keeps a constant in most of the tube center regions if the ratio of the tube diameter to the Debye length is larger than about 10.

For EOF in annular tubes, we find that both the electric potential distribution and the velocity are influenced by the ratio of inner radius to outer radius $R_1/R_2$, the value of $\psi_1/\psi_2$, and the Debye length. For the electro-osmotic pulsating flow in annular tubes, the ratio of the annulus gap to Debye length has a significant effect on the velocity amplitude and most of the bulk fluid is unperturbed if this ratio is large. When the frequency is small, the velocity perturbation propagates from the near-wall region deeper into the bulk region. However, for the cases of large frequency, the perturbed area is restricted within a thin layer very close to the radius wall.

The non-Newtonian effect on EOF has also been investigated. For electro-osmotic driven non-Newtonian fluid flow in circular pipes, our investigation shows that the velocity
profile is dominated by the fluid rheological behavior. The shear thickening fluid is more difficult to grow into a pluglike flow than Newtonian. In contrast, the shear thinning fluid is more inclined to develop into the pluglike flow. For the flow in annular ducts, the velocity profile is significantly dependent on the ratio of inner radius to outer radius $R_1/R_2$ as well. The present results demonstrate that the present axisymmetric lattice Boltzmann method can be used as an effective numerical tool for studying the EOF in annular microducts. The results obtained from this work not only can provide insights into the EOF but also can aid in predicting EOF mass flow rate or species mixing in practical applications of electrokinetic micropumps or micromixers.

ACKNOWLEDGMENTS

The authors would like to thank the National Natural Science Foundation of China for the support under Grant No. 50776067, the Science and Technology Research Development Program of Shaanxi Province for the support under Grant No. 2010KJXX-01, and the Program for NCET in university for the support under the Grant no. NCET-07-0676.

APPENDIX: RECOVERING THE MACROSCOPIC PB EQUATION THROUGH THE CHAPMAN–ENSKOG PROCEDURE

In the appendix, we present the detailed mathematical derivation of Eq. (4) from Eq. (17) through the Chapman–Enskog procedure. In order to recover Eq. (4), we first introduce the following evolution equation:

$$g_a(x + c_a \delta_1 t + \delta_1) - g_a(x, t)$$

$$= - \frac{1}{2 \tau_g} [g_a(x + c_a \delta_1 t + \delta_1) - g_{eq}(x + c_a \delta_1 t + \delta_1)]$$

$$- \frac{1}{2 \tau_g} [g_a(x, t) - g_{eq}(x, t)]$$

$$- \frac{c_{ar}}{r} \delta_1 [g_a(x, t) - g_{eq}(x, t)]$$

$$+ \frac{\delta_1}{2} [S_i(x + c_a \delta_1 t + \delta_1) - S_i(x, t)],$$

(A1)

where $g_a$ is the electric potential distribution function, $\tau_g$ is nondimensional relaxation time, $S_i$ is source term, $c_a = (c_{ax}, c_{ay})$ is the discrete velocities, and $g_{eq}$ is the corresponding equilibrium distribution function, with $g_{eq}^{0} = 4 \psi/9$.
produced to Taylor-series expansion, the evolution Eq. (A1) is reduced to
\[
\delta_t \left( \partial_t + c_a \cdot \nabla \right) g_{\alpha} + \frac{\delta_t^2}{2} \left( \partial_t + c_a \cdot \nabla \right)^2 g_{\alpha} = - \frac{1}{\tau_g} \left( g_{\alpha} - g_{\alpha}^{eq} \right)
\]
\[
- \frac{\delta_t}{2\tau_g} \left( \partial_t + c_a \cdot \nabla \right) \left( g_{\alpha} - g_{\alpha}^{eq} \right) - \frac{c_{ar}}{r} \delta_t \left( g_{\alpha} - g_{\alpha}^{eq} \right) + \delta S_1
\]
\[
+ \frac{\delta_t^2}{2} \left( \partial_t + c_a \cdot \nabla \right) S_1.
\]
Introducing the following multiscale expansions:
\[
\partial_t = \partial_{\phi} + \delta_t \partial_{\phi}, \quad g_a = g_a^{(0)} + \delta g_a^{(1)} + \delta^2 g_a^{(2)},
\]
and substituting Eq. (A4) into Eq. (A3), Eq. (A3) to order \( (\delta_t)^0 \) is
\[
g_a^{(0)} = g_{\alpha}^{eq},
\]
to order \( (\delta_t)^1 \) is
\[
(\partial_{\phi} + c_a \cdot \nabla) g_a^{(0)} + \frac{1}{\tau_g} g_a^{(1)} = S_1,
\]
and to order \( (\delta_t)^2 \) is
\[
(\partial_{\phi} + c_a \cdot \nabla) g_a^{(0)} + \frac{1}{2} \left( \partial_{\phi} + c_a \cdot \nabla \right) g_a^{(0)} + \frac{1}{\tau_g} g_a^{(2)}
\]
\[
+ \frac{1}{2\tau_g} \left( \partial_{\phi} + c_a \cdot \nabla \right) g_a^{(1)} = - \frac{c_{ar}}{r} g_a^{(1)}
\]
\[
+ \frac{1}{2} \left( \partial_{\phi} + c_a \cdot \nabla \right) S_1.
\]
Applying Eq. (A6), we can rewrite Eq. (A7) as
\[
(\partial_{\phi} + c_a \cdot \nabla) g_a^{(0)} + \frac{1}{\tau_g} g_a^{(2)} = - \frac{c_{ar}}{r} g_a^{(1)}.
\]
Taking the summations of Eqs. (A6) and (A8), we can obtain
\[
\partial_{\phi} \psi = \sum_a S_1,
\]
\[
\partial_{\phi} \psi + \partial_t \left( \sum_a c_{ar} g_a^{(1)} \right) = - \frac{1}{\tau_g} \sum_a c_{ar} g_a^{(1)}.
\]
In order to recover the macroscopic Eq. (4), \( \Sigma_a S_1 \) should be given by \( \Sigma_a S_1 = -(2n_\alpha e / \varepsilon e_0) \sinh (ze/\psi / k_B T) \), and we have
\[
S_1 = - \frac{2n_\alpha e}{9\varepsilon e_0} \sinh \left( ze/\psi / k_B T \right),
\]
where \( \psi \) is the value of the previous iteration step, so \( S_1 \) is a determined value.

From Eq. (A6), we have
\[
\sum_a c_{ar} g_a^{(1)} = - \tau_g \delta_t \sum_a c_{ar} g_a^{(0)}.
\]
From Eqs. (A2) and (A5), we can obtain
\[
\partial_t \sum_a c_{ar} g_a^{(0)} = c_t \partial_t \psi.
\]
Therefore, Eq. (A12) can be reduced to

FIG. 14. Normalized electro-osmotic velocity profiles for power-law flow in annular ducts. (a) \( R_1 / R_2 = 0.5, c_a = 10^{-5} \text{ M} \). (b) \( R_1 / R_2 = 0.5, c_a = 10^{-5} \text{ M} \). (c) \( R_1 / R_2 = 0.09, c_a = 10^{-5} \text{ M} \).

\[ g_{\alpha}^{eq} = \psi/9, g_{\alpha}^{eq} = \psi / 36. \] It can be found that
\[
\sum_a g_{\alpha}^{eq} = \psi, \quad \sum_a c_{ar} g_{\alpha}^{eq} = 0, \quad \sum_a c_{ar} c_{ar} g_{\alpha}^{eq} = c^2 t^2 \psi \delta \psi.
\]
\[ \sum_{a} c_{a} g^{(1)}_{a} = -\tau_{g} \frac{\partial}{\partial t} \psi. \]  

(A14)

Substituting Eq. (A14) into Eq. (A10) and then combing Eq. (A9) with Eq. (A10), we can obtain the following macroscopic electric potential equation:

\[ \tau_{g} \frac{\partial}{\partial t} \psi + \frac{1}{r} \frac{d}{dr} \left( r \tau_{g} \frac{d\psi}{dr} \right) = \frac{2n_{e} z e}{\varepsilon \varepsilon_{0}} \sinh \left( \frac{z e \psi}{k_{B} T} \right), \]  

(A15)

where the potential diffusivity \( \chi = \delta \tau_{g} \frac{e^{2}}{\varepsilon} \). If assuming that the steady-state pure electrical driven flow is fully developed and the potential diffusivity \( \chi \) is equal to unity, Eq. (A15) can be reduced to

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\psi}{dr} \right) = \frac{2n_{e} z e}{\varepsilon \varepsilon_{0}} \sinh \left( \frac{z e \psi}{k_{B} T} \right). \]  

(A16)

To eliminate the implicitness of Eq. (A1), a distribution function \( g_{a} = g_{a} + 0.5 \left( g_{a} - g_{aq}^{\Omega} / \tau_{g} - 0.5 \delta S_{1} \right) \) can be shown. Through some standard algebra, the evolution equation for \( \bar{g}_{a} \) can be obtained,

\[ \bar{g}_{a}(\mathbf{x} + c_{a} \delta_{i} t + \delta_{i}) - \bar{g}_{a}(\mathbf{x}, t) = -\omega_{g} [\bar{g}_{a}(\mathbf{x}, t) - g_{aq}^{\Omega}(\mathbf{x}, t)] \]

\[ + (1 - 0.5 \omega_{g}) \delta S_{1}(\mathbf{x}, t). \]  

(A17)

where \( \omega_{g} \) is given by \( \omega_{g} = [1 + (c_{a} \tau_{g} / r)] / (\tau_{g} + 0.5) \). The macroscopic electric potential \( \psi \) can be calculated with an iteration method as

\[ \psi = \sum_{a} \bar{g}_{a} - 0.5 \delta_{i} \frac{2n_{e} z e}{\varepsilon \varepsilon_{0}} \sinh \left( \frac{z e \psi}{k_{B} T} \right). \]  

(A18)

For simplicity, we use \( g \) instead of \( \bar{g} \) in the paper.

\[ ^{1}D. \ Q. \ Li, \ Electroknetics \ in \ Microfluidics \ (Elsevier \ Academic, \ Boston, \ 2004). \]

\[ ^{2}C. \ M. \ Hou, \ Microflows \ and \ Nanoflows \ Fundamentals \ and \ Simulation \ (Springer, \ New \ York, \ 2005). \]

\[ ^{3}J. \ K. \ Wang, \ M. \ R. \ Wang, \ and \ Z. \ X. \ Li, \ J. \ Colloid \ Interface \ Sci. \ 296, \ 729 \ (2006). \]

\[ ^{4}L. \ Q. \ Ren \ and \ D. \ Q. \ Li, \ J. \ Colloid \ Interface \ Sci. \ 243, \ 255 \ (2001). \]

\[ ^{5}Y. \ H. \ Zhang, \ X. \ J. \ Gu, \ R. \ W. \ Barber, \ and \ D. \ R. \ Emerson, \ J. \ Colloid \ Interface \ Sci. \ 275, \ 670 \ (2004). \]

\[ ^{6}Z. \ L. \ Guo, \ T. \ S. \ Zhao, \ and \ Y. \ Shi, \ J. \ Chem. \ Phys. \ 122, \ 144907 \ (2005). \]

\[ ^{7}G. \ H. \ Tang, \ Z. \ L. \ Guo, \ Y. \ L. \ He, \ and \ W. \ Q. \ Tao, \ J. \ Appl. \ Phys. \ 100, \ 094908 \ (2006). \]

\[ ^{8}Z. \ H. \ Chai, \ Z. \ L. \ Guo, \ and \ B. \ C. \ Shi, \ J. \ Appl. \ Phys. \ 101, \ 104913 \ (2007). \]

\[ ^{9}J. \ Wu, \ J. \ Appl. \ Phys. \ 103, \ 024907 \ (2008). \]

\[ ^{10}G. \ H. \ Tang, \ Y. \ L. \ He, \ and \ W. \ Q. \ Tao, \ J. \ Appl. \ Phys. \ 107, \ 104906 \ (2010). \]

\[ ^{11}S. \ Chen \ and \ G. \ Doolen, \ Annu. \ Rev. \ Fluid \ Mech. \ 30, \ 329 \ (1998). \]

\[ ^{12}P. \ B. \ Warren, \ Int. \ J. \ Mod. \ Phys. \ C, \ 8, \ 889 \ (1997). \]

\[ ^{13}B. \ M. \ Li \ and \ D. \ Y. \ Kwok, \ J. \ Chem. \ Phys. \ 120, \ 947 \ (2004). \]

\[ ^{14}Z. \ Tian \ and \ D. \ Y. \ Kwok, \ Langmuir \ 21, \ 2192 \ (2005). \]

\[ ^{15}S. \ Melchionna \ and \ S. \ucci, \ J. \ Chem. \ Phys. \ 120, \ 4492 \ (2004). \]

\[ ^{16}F. \ Capuani, \ S. \ Pagonabarraga, \ and \ D. \ Frenkel, \ J. \ Chem. \ Phys. \ 121, \ 973 \ (2004). \]

\[ ^{17}W. \ J. \ Kim, \ H. \ G. \ Chen, \ and \ M. \ S. \ Jhon, \ J. \ Appl. \ Phys. \ 99, \ 08N106 \ (2006). \]

\[ ^{18}G. \ H. \ Tang, \ X. \ F. \ Li, \ Y. \ L. \ He, \ and \ W. \ Q. \ Tao, \ J. \ Non-Newtonian \ Fluid \ Mech. \ 157, \ 133 \ (2009). \]

\[ ^{19}J. \ Halliday, \ L. \ A. \ Hammond, \ C. \ M. \ Care, \ K. \ Good, \ and \ A. \ Stevens, \ Phys. \ Rev. \ E \ 64, \ 011208 \ (2001). \]

\[ ^{20}T. \ Lee, \ H. \ Huang, \ and \ C. \ Shu, \ Int. \ J. \ Mod. \ Phys. \ C \ 17, \ 645 \ (2006). \]

\[ ^{21}H. \ Huang, \ T. \ Lee, \ and \ C. \ Shu, \ Int. \ J. \ Numer. \ Methods \ Heat Fluid \ Flow \ 17, \ 587 \ (2007). \]

\[ ^{22}T. \ Reis \ and \ T. \ N. \ Phillips, \ Phys. \ Rev. \ E \ 75, \ 056703 \ (2007). \]

\[ ^{23}T. \ Reis \ and \ T. \ N. \ Phillips, \ Phys. \ Rev. \ E \ 77, \ 026703 \ (2008). \]

\[ ^{24}J. \ G. \ Zhou, \ Phys. \ Rev. \ E \ 78, \ 036701 \ (2008). \]

\[ ^{25}S. \ Chen, \ J. \ Tolke, \ S. \ Geller, \ and \ M. \ Krafczyk, \ Phys. \ Rev. \ E \ 78, \ 046703 \ (2008). \]

\[ ^{26}Z. \ L. \ Guo, \ H. \ F. \ Han, \ B. \ C. \ Shi, \ and \ C. \ G. \ Zheng, \ Phys. \ Rev. \ E \ 79, \ 046708 \ (2009). \]

\[ ^{27}Q. \ Li, \ Y. \ L. \ He, \ G. \ H. \ Tang, \ and \ W. \ Q. \ Tao, \ Phys. \ Rev. \ E \ 80, \ 037702 \ (2009). \]

\[ ^{28}H. \ B. \ Huang \ and \ X. \ Y. \ Lu, \ Phys. \ Rev. \ E \ 80, \ 016701 \ (2009). \]

\[ ^{29}S. \ Chakraborty \ and \ S. \ Ray, \ Phys. \ Fluids \ 20, \ 083602 \ (2008). \]

\[ ^{30}C. \ Rice \ and \ R. \ Whitehead, \ J. \ Phys. \ Chem. \ 69, \ 4017 \ (1965). \]

\[ ^{31}Q. \ Zou \ and \ X. \ He, \ Phys. \ Fluids \ 9, \ 1591 \ (1997). \]

\[ ^{32}G. \ H. \ Tang, \ W. \ Q. \ Tao, \ and \ Y. \ L. \ He, \ Phys. \ Rev. \ E \ 72, \ 016703 \ (2005). \]

\[ ^{33}R. \ C. \ Wu \ and \ K. \ D. \ Papadopoulos, \ Colloids \ Surf., \ A \ 161, \ 469 \ (2000). \]