Technical Note

Analytical solution for sound radiated from the rotating point source in uniform subsonic axial flow

Yijun Mao *, Chen Xu, Datong Qi

School of Energy and Power Engineering, Xi’an Jiaotong University, Xi’an, Shaanxi 710049, PR China

A R T I C L E   I N F O

Article history:
Received 3 July 2014
Received in revised form 24 December 2014
Accepted 29 December 2014

Keywords:
Aeroacoustics
Rotating source
Moving medium
Exact solution
Series expansion method

A B S T R A C T

The paper presents an analytical solution for sound radiated from the rotating monopole and dipole point sources in uniform subsonic axial flow by employing the spherical harmonic series expansion method. The analytical results obtained from the present series expansion formulations have a good agreement with the numerical results obtained from the time-domain and frequency-domain numerical methods. The results also indicate that the effect of the freestream Mach number on the directivity pattern is various for different types of the rotating sources.

© 2015 Elsevier Ltd. All rights reserved.

1. Introduction

Acoustic characteristics of the rotating sources have been the research interest for the past decades. Many acoustic prediction methods, such as the time-domain numerical method (TDNM) [1,2], the frequency-domain numerical method (FDNM) [3–6] and the spherical harmonic series expansion method (SHSEM) [7–10], have been developed to predict the sound radiated from the rotating sources. Concerning the SHSEM, it not only avoids interpolation error, the singular integral and the sophisticated algorithm for the retarded-time equation, but also provides an analytical solution for sound radiated from the rotating point source. A recent review on the above methods can be found in Refs. [5,8]. In these methods, one common assumption is that the medium is quiescent for acoustic computation.

In engineering applications, cases of moving medium are often encountered. Some examples are the rotating blades operating at the moving fluid, such as the axial flow fan, the underwater propeller and the open rotor aircraft engine. The background flow can affect not only the source strength but also the sound propagation. The effect of uniform flow on the source strength can be simulated by using the computational fluid dynamics method and its effect on the sound propagation can also be taken into account by employing the convective Green’s function. Recently, the time-domain acoustic pressure formulation for the sources in uniform subsonic flow was proposed by Najafi-Yazdi et al. [11] and Ghorbaniasl and Lacor [12]. After that, Xu et al. [13] developed the frequency-domain acoustic pressure formulation for the rotating sources in uniform subsonic flow. An alternative acoustic prediction method, the series expansion solutions for the acoustic pressure of the rotating monopole and dipole point sources in uniform subsonic axial flow are derived in the paper.

The layout of this note is as follows. In Section 2, the coordinate system and the Green’s function for the convective wave equation are introduced. In Section 3, the series expansion solutions for the rotating monopole and dipole point sources in uniform subsonic axial flow are derived. The numerical validation of the proposed formulations is carried out and the effect of the flow Mach number on the directivity pattern is analyzed in Section 4. Section 5 draws the conclusion.

2. Coordinate system and Green’s function

2.1. Description of coordinate system and related parameters

As defined in the previous literature [8], not only the Cartesian coordinate system but also both the cylindrical and spherical coordinate systems are established to describe the sound radiation from the rotating monopole and dipole point sources. As shown in Fig. 1, the position vector of the stationary observer is \( x(x_1, x_2, x_3) \), \( x(R_x, Z_1, \phi_z) \) and \( x(r_x, \theta, \phi) \) in the Cartesian, cylindrical and spherical coordinate systems, respectively. The position vector of the rotating point source is \( y(y_1, y_2, y_3) \), \( y(R_y, Z_2, \phi_y) \) and
Nomenclature

- $c_0$: speed of sound in unperturbed medium, m s$^{-1}$
- $g$: time-domain Green's function
- $G$: frequency-domain Green's function
- $h_n^{(1)}(\cdot)$: spherical Hankel function of the first kind and order $n$
- $j_n(\cdot)$: spherical Bessel function of order $n$
- $L$: strength of dipole source, N
- $m$: harmonic number
- $N_r$: truncation number of infinite series
- $p$: acoustic pressure, Pa
- $P_n^m(\cdot)$: associated Legendre function of the first kind of degree $n$ and order $m$
- $Q$: strength of monopole source, kg s$^{-1}$
- $r$: distance between source and receiver, $|x - y|$, m
- $t$: observer time, s
- $U_0$: freestream velocity, m s$^{-1}$
- $x$: observer position vector, m
- $y$: source position vector, m
- $\phi$: azimuth angle, rad
- $\theta$: elevation angle, rad
- $\tau$: source time, s
- $\omega_R$: angular frequency of source rotation, rad s$^{-1}$
- $\omega_0$: angular frequency of source pulsation, rad s$^{-1}$

Subscripts

- $x$: observer quantity
- $y$: source quantity

Abbreviations

- TDNM: time-domain numerical method
- FDNM: frequency-domain numerical method
- RMPS: rotating monopole point source
- RRDP: rotating radial dipole point source
- RCDPS: rotating circumferential dipole point source
- RADPS: rotating axial dipole point source

2.2. Green's function for uniform inflow

The convective wave equation with the monopole and dipole point sources is as follows:

$$\left(\frac{1}{c_0^2} \frac{D^2}{Dt^2} - \nabla^2\right) p(x, t) = \frac{D}{Dt} \left[ Q(x, t) \delta(x - y) \right] - \frac{\partial}{\partial \theta_i} \left[ L_i(x, t) \delta(x - y) \right]$$

with

$$D = \frac{\partial}{\partial t} + U_0 \cdot \nabla$$

where $t$ is the observer time, $c_0$ is the sound speed, $U_0$ is the constant velocity of the uniform flow, $p$ is the time-domain acoustic pressure, $\delta$ is the Dirac delta function, and $Q$ and $L$ are, respectively, the instantaneous source strength of the monopole and dipole point sources. In the present paper, it is assumed that the subsonic flow is along the positive $Z$ direction, and the three-dimensional free-space time-domain Green's function for the convective wave equation is [11]

$$g(x, y, t - \tau) = \frac{1}{4\pi r} \delta \left[ t - \tau + \left( \frac{\vec{r} \cdot \vec{M}_0 \vec{r}_1}{c_0^2 \beta} \right) \right]$$

with

$$\vec{r} = \sqrt{\beta^2 (y_1(\tau) - x_1)^2 + \beta^2 (y_2(\tau) - x_2)^2 + (y_3 - x_3)^2}$$

where $\vec{r}_1 = x_1 - y_1, M_0 = |U_0|/c_0, \beta = \sqrt{1 - \frac{c_0^2}{M_0^2}}$, and the symbol $\vec{r}$ over the quantity means the quantity related to the uniform flow.

By employing the definition of the Fourier transform, we can derive the frequency-domain Green's function as follows

$$\tilde{g}(\mathbf{x}, \mathbf{y}, \omega) = \int_{-\infty}^{\infty} g(\mathbf{x}, \mathbf{y}, t - \tau) e^{i\omega(t - \tau)} dt = \frac{\epsilon^{ik \cdot \vec{r}} \epsilon^{-i\omega_0 M_0 \vec{r}}}{4\pi \dot{r}}$$

where $\vec{k} = k / \beta^2$ and $k = \omega / c_0$.

The position vector related to the stationary observer in a uniform axial flow is defined as follows

$$\vec{x} = (\beta x_1, \beta x_2, x_3) = (\tilde{r}_x, \tilde{\theta}_x, \tilde{\phi}_x) = (\tilde{R}_x, \tilde{Z}_x, \tilde{\phi}_x)$$

with

$$\tilde{r}_x = \sqrt{\beta^2 x_1^2 + \beta^2 x_2^2 + x_3^2}, \quad \cos \tilde{\theta}_x = x_3 / \tilde{r}_x, \quad \tilde{\phi}_x = \phi_x, \quad \tilde{R}_x = \beta \sqrt{x_1^2 + x_2^2}, \quad \tilde{Z}_x = Z_x = x_3.$$

Similarly, the position vector related to the rotating source in uniform axial flow can be expressed as follows

$$\vec{y} = (\beta y_1(\tau), \beta y_2(\tau), y_3(\tau)) = (\tilde{r}_y, \tilde{\theta}_y, \tilde{\phi}_y(\tau)) = (\tilde{R}_y, \tilde{Z}_y, \tilde{\phi}_y(\tau))$$

with

$$\tilde{r}_y = \sqrt{\beta^2 y_1^2 + \beta^2 y_2^2 + y_3^2}, \quad \cos \tilde{\theta}_y = y_3 / \tilde{r}_y, \quad \tilde{\phi}_y(\tau) = \phi_y(\tau) = \phi_0 + \omega_R \tau, \quad \tilde{R}_y = \beta \sqrt{y_1^2 + y_2^2}, \quad \tilde{Z}_y = Z_y = y_3.$$

Based on the above definitions, we can deduce the following spherical harmonic series expansion expression:

$$\frac{\epsilon^{ik \cdot \vec{r}}}{4\pi \dot{r}} = \frac{i \tilde{k}}{4\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_m \tilde{c}_{1nm} \tilde{a}_{1m}$$
with

\[ e_m = \begin{cases} 0.5 & m = 0 \\ 1 & m \neq 0 \end{cases} \] (12)

\[ a_{1m} = e^{i(m_0 - \phi)} + e^{-i(m_0 - \phi)} \] (13)

\[ b_{1m} = \int f(x) h_0^1(\hat{r}) h_0^1(\hat{r}) \hat{r} \hat{r} \] (14)

\[ c_{1mm} = \frac{(n - m)!}{(n + m)!} \hat{P}_n^m(\cos \hat{\theta}) \hat{P}_{m+1}^m(0) \] (15)

where \( f(x) \) and \( h_0^1 \) are the first kind of spherical Bessel function and Hankel function, respectively, \( \hat{P}_n^m(\cdot) \) is the associated Legendre function of the first kind of degree \( n \) and order \( m \). Note that only the term \( a_{1m} \) in Eq. (11) is dependent on the source point \( \tau \), and Eq. (11) will be used in the following section to deduce the series expansion solutions for the sound radiation from the rotating monopole and dipole sources in a uniform subsonic axial flow.

### 3. Series expansion solution for rotating source in uniform axial inflow

#### 3.1. Rotating monopole point source

Assuming that the monopole point source rotates at a constant speed, we can deduce the frequency-domain acoustic pressure formulation as follows [13]

\[
p_{TM}(x, \omega) = \left( -i \omega + U_0 \frac{\partial}{\partial Z} \right) \int_{-\infty}^{\infty} Q(y, \tau) G(x, y, \omega) e^{i\tau\omega} \, d\tau
\] (16)

where the symbol \( \sim \) means the frequency-domain complex quantity. The partial derivative of the Green’s function in the axial direction

\[
- \frac{\partial G(x, y, \omega)}{\partial Z} = \frac{\partial G(x, y, \omega)}{\partial Z} = - \frac{M_0 e^{i(k \tau \cdot \hat{r} + \hat{r} \cdot \hat{r})}}{4\pi r} \sum_{n=0}^{\infty} (2n + 1) b_{1n} \sum_{m=0}^{n} e_{1mn} a_{1m}
\] (17)

with

\[ e_{1mn} = \frac{(n - m + 1)!}{(n + m)!} \hat{P}_n^m(\cos \hat{\theta}) \hat{P}_{m+1}^m(0) \] (18)

Moreover, by assuming that the rotating monopole point source (RMPS) is monochromatic, we have

\[ Q(\omega_0)\delta(\omega - \omega_0) = \int_{-\infty}^{\infty} Q(\tau) e^{i\tau\omega} \, d\tau \] (19)

where \( \omega_0 \) and \( Q \) are the angular frequency and complex strength of the RMPS. Substituting Eqs. (11), (17) and (19) into Eq. (16), we can obtain the series expansion solution for the RMPS as follows

\[
p_{TM}(x, \omega) = \frac{\omega M_0 e^{i(k \tau \cdot \hat{r} + \hat{r} \cdot \hat{r})}}{4\pi r} \sum_{n=0}^{\infty} (2n + 1) b_{1n} \sum_{m=0}^{n} e_{1mn} a_{1m} \] (20)

with

\[ d_{1m} = e^{i(m_0 - \phi)} \delta(\omega - m\omega_0 - \omega_0) + e^{-i(m_0 - \phi)} \delta(\omega - m\omega_0 - \omega_0) \] (21)

Because Eq. (21) implies that the sound radiated from the rotating monochromatic point source is tonal, Eq. (20) can be simplified as follows

\[
p_{TM}(x, \omega_0) = \frac{\omega M_0 e^{i(k \tau \cdot \hat{r} + \hat{r} \cdot \hat{r})}}{4\pi r} \sum_{n=0}^{\infty} (2n + 1) b_{1n} \sum_{m=0}^{n} e_{1mn} a_{1m} \] (22)

with

\[ e_{1mn} = \frac{(n - m)!}{(n + m)!} \hat{P}_n^m(\cos \hat{\theta}) \hat{P}_{m+1}^m(0) \] (23)

for the monochromatic dipole source with the angular frequency \( \omega_0 \), we have

\[ L_i(\omega_0) = \int_{-\infty}^{\infty} L_i(\tau) e^{i\tau\omega} \, d\tau \] (24)

for the monochromatic dipole source with the angular frequency \( \omega_0 \), we have

\[ L_i(\omega_0) = \int_{-\infty}^{\infty} L_i(\tau) e^{i\tau\omega} \, d\tau \] (25)

where \( L_i(\omega_0) \) is the component of the dipole source strength in the ith direction.

By substituting Eqs. (17), (27)–(30) into Eq. (25), we can deduce the total acoustic pressure of the noise radiated from the rotating dipole point source as follows

#### 3.2. Rotating dipole point source

The frequency-domain acoustic pressure of the noise radiated from the rotating dipole point source in the uniform axial flow can be expressed as [13]

\[
p_{TD}(x, \omega) = \int_{-\infty}^{\infty} L(x, \tau) \cdot \nabla G(x, y, \omega) \, e^{i\tau\omega} \, d\tau
\] (26)

Moreover, the gradient of the Green’s function in the cylindrical coordinate system is

\[
- \frac{\partial G(x, y, \omega)}{\partial Z} = \frac{\partial G(x, y, \omega)}{\partial Z} = - \frac{k e^{i(k \tau \cdot \hat{r} + \hat{r} \cdot \hat{r})}}{4\pi r} \sum_{n=0}^{\infty} (2n + 1) b_{1n} \sum_{m=0}^{n} e_{1mn} a_{1m}
\] (27)

\[
- \frac{\partial G(x, y, \omega)}{\partial \phi} = - \frac{k e^{i(k \tau \cdot \hat{r} + \hat{r} \cdot \hat{r})}}{4\pi r} \sum_{n=0}^{\infty} (2n + 1) b_{1n} \sum_{m=0}^{n} e_{1mn} a_{2m}
\] (28)

with

\[ a_{2m} = e^{i(m_0 - \phi)} \delta(\omega - m\omega_0 - \omega_0) + e^{-i(m_0 - \phi)} \delta(\omega - m\omega_0 - \omega_0) \] (29)

For the monochromatic dipole source with the angular frequency \( \omega_0 \), we have

\[ L_i(\omega_0) = \int_{-\infty}^{\infty} L_i(\tau) e^{i\tau\omega} \, d\tau \] (30)

where \( L_i(\omega_0) \) is the component of the dipole source strength in the ith direction.
\[ \bar{p}_{DA}(x, \omega_m) = \bar{p}_{DM}(x, \omega_m) + \bar{p}_{DB}(x, \omega_m) + \bar{p}_{DM}(x, \omega_m) \]  

with

\[ \bar{p}_{DM}(x, \omega_m) = \frac{i\mu k^2 L_x(\omega_m)}{4\pi} e^{-i k M_r x} e^{-i m \phi_0} \sum_{n=0}^{\infty} \frac{1}{2n+1} \tilde{b}_{(n+1)} \tilde{c}_{3mn} \]  

\[ \bar{p}_{DB}(x, \omega_m) = \frac{\tilde{m} k L_y(\omega_m)}{4\pi} e^{-i k M_r x} e^{-i m \phi_0} \sum_{n=0}^{\infty} (2n+1) \tilde{b}_{(n+1)} \tilde{c}_{3mn} \times \tilde{c}_{3mn} \]  

\[ \bar{p}_{DM}(x, \omega_m) = \frac{-i k L_z(\omega_m)}{4\pi} e^{-i k M_r x} e^{-i m \phi_0} \sum_{n=0}^{\infty} (2n+1) \tilde{b}_{(n+1)} \tilde{c}_{3mn} \]  

Eqs. (32)–(34) represent the tonal sound radiated from the radial, circumferential and axial dipole point sources, respectively. Eqs. (22), (32)–(34) are the main results of the present paper, in which the effect of the uniform subsonic axial inflow on the sound propagation is explicitly taken into account. Compared with the series expansion solutions for the rotating sources in a quiescent medium [8], it is observed that the terms not in the first line of Eqs. (22), (32)–(34) are the additional terms due to the uniform flow. The uniform axial flow also affects the phase of the complex acoustic pressure, by the term \( e^{-i k M_r x} \) in each source component. Additionally, the azimuth angle \( \phi_0 \) only affects the phase of the complex acoustic pressure, and the same conclusion is also drawn from the rotating point sources in quiescent medium [8]. Because of the above acoustic radiation feature, only the relationship between the directivity pattern (DP) and the elevation angle \( \phi_0 \) is analyzed in the following section.

Moreover, the truncation number for Eqs. (22), (32)–(34) should be analyzed. In the previous papers concerning the series expansion solution for the rotating point sources in quiescent medium, it has been concluded that the truncation number \( N_{tr} \) should meet the following requirement [7,8]

\[ N_{tr} \geq \max(|m|, k \min(r_x, r_y)) \]  

where max and min mean the relatively larger and smaller number, respectively. Since Eqs. (22), (32)–(34) are similar to the series expansion expressions for the quiescent medium, the corresponding truncation number \( \tilde{N}_{tr} \) is as follows

\[ \tilde{N}_{tr} \geq \max(|m|, \tilde{k} \min(r_x, r_y)) \]  

4. Numerical validation and discussion

In this section, the aforementioned series expansion formulations for the rotating monopole and dipole point sources are validated by comparing the analytical results with the numerical results obtained from the time-domain formulation of Najafi-Vazdi et al. [11] and Ghobanosal and Lacor [12] and the frequency-domain formulation of Xu et al. [13]. With the above three methods, the acoustic pressure spectra for the rotating monopole and dipole point sources are computed, respectively. In both the test cases, the speed of sound is \( c_0 = 340 \text{ m/s} \), and the Mach number of the uniform axial flow is \( M_0 = 0.5 \). The pulsating frequency and the rotational frequency of the point source are equal to \( f_p = 150 \text{ Hz} \) and \( f_R = 30 \text{ Hz} \), respectively. The point source rotates counter-clockwise around the \( Z \) axis at the radius \( r_y = 1 \text{ m} \) on the \( XY \) plane and the initial azimuthal angle is \( \phi_0 = 0 \). The stationary observer is located on the plane of source rotation with \( r_x = 2 \text{ m} \) and \( \phi_r = Z_s = 0 \). The detailed introduction on the numerical treatment of the TDNM and the FDNM is given in Ref. [13].

Fig. 2 shows the acoustic pressure spectrum for the RMPS with the unit source strength, i.e., \( Q = 1 \text{ kg s}^{-1} \). Figs. 3–5 display the acoustic pressure spectra for the rotating dipole point source with the unit source strength in the radial, circumferential and axial directions, respectively. The abbreviations of RRDPs, RCDPs and RADPS represent the rotating dipole point source in the radial, circumferential and axial directions, respectively. For both the rotating monopole and dipole point sources, the results from the three different methods are in good agreement with each other, thus validating the present series expansion solutions for the uniform subsonic axial flow.

In the following, the effect of the flow Mach number \( M_0 \) on the DP is analyzed for the rotating monopole and dipole point sources. The pulsating frequency, the rotational frequency and radius of the source are the same as those for the previous test case. The acoustic pressure at three different flow Mach numbers, i.e., \( M_0 = 0, 0.4 \) and 0.8, is illustrated in Fig. 4. The acoustic pressure spectrum for the RRDPs is shown in Fig. 5.
and 0.8, are computed with the proposed series expansion formulation. As mentioned in Section 3, the amplitude of the acoustic pressure is independent of the azimuth angle \( \phi_x \). Here we define the DP as

\[
D(\theta_x, M_0) = \frac{L_p(\theta_x, M_0)}{\max(L_p(\theta_x, M_0))}
\]

(38)

with

\[
L_p(\theta_x, M_0) = 20 \log_{10} \left( \sqrt{\sum_{m=-m_{\text{max}}}^{m_{\text{max}}} |p(\theta_x, \omega_m)|^2} \times 10^{-5} \right)
\]

(39)

where \( L_p \) is the overall acoustic pressure level of the local observer. \( m_{\text{max}} \) and \( m_{\text{min}} \) are, respectively, the maximum and minimum of the harmonic number \( m \). In the present computation with \( m_{\text{max}} = 25 \) and \( m_{\text{min}} = -4 \), it is accurate enough to compute the overall acoustic pressure level according to the acoustic pressure spectra in Figs. 1–4.

Figs. 5–9 illustrate the effects of the flow Mach number on the DP for the rotating monopole and dipole point sources, respectively. For the RMPS, the uniform axial flow makes the acoustic pressure level upstream of the source is greater than that downstream of the source. The similar phenomenon has also been observed in Fig. 9 for the RADPS. For the RRDPS and RCDPS, the higher the flow Mach number is, the more percent of the acoustic energy radiates on the plane of source rotation. However, the DP of the RRDPS and RCDPS is symmetrical on the plane of source rotation even at the case that the uniform flow is along the axial direction.

The above phenomenon can be explained from the analytical formulations (22), (32), (33) and (34). In the above analytical formulations, the elevation angle \( \theta_x \) is included in both the terms \( c_{3mn} \) and \( e_{3mn} \). The analytical formulations for the RRDPS and RCDPS only contain the term \( c_{3mn} \), whereas the analytical formulations for the RMPS and RADPS are also related to the term \( e_{3mn} \). Moreover, the associated Legendre function has the following characteristic [14]

\[
P_n^{(m)}(0) \neq 0, \ n = |m| + 2l
\]

(40)

where \( l \) is an integer not less than zero. By substituting Eq. (40) into the Eqs. (23) and (24), we can obtain

\[
c_{3mn} \propto P_n^{(m+2)}(\cos \theta_x) \equiv P_n^{(m+2)}(-\cos \theta_x)
\]

(41)

\[
e_{3mn} \propto P_n^{(m+2)}(\cos \theta_x) \equiv -P_n^{(m+2)}(-\cos \theta_x)
\]

(42)

where \( \propto \) is the proportional symbol. From Eqs. (41) and (42), we can conclude that, in the uniform axial flow, the DPs for the RRDPS and RCDPS are symmetrical on the plane of source rotation whereas the DPs for the RMPS and RADPS are asymmetrical. The above feature is applied to analyze the effect of uniform axial flow on noise propagation of the axial flow rotor in subsonic rotation. Since the
noise is mainly generated from pressure fluctuation on the blade surface and the components in axial and circumferential directions are usually predominant, we can deduce from Figs. 8 and 9 that the acoustic energy for axial flow rotor inclines to propagation along the upstream direction. Note that the present analytical method is only suitable for the case of axial uniform flow. An alternative FDNM for predicting noise radiated from the rotating source in the uniform flow with arbitrary direction can be found in reference [13].

5. Conclusion

In this paper, the analytical series expansion solutions for predicting the sound radiated from the rotating monopole and dipole sources have been extended. The extended solution can take into account the effect of the uniform subsonic axial flow on the sound propagation. Its validation was demonstrated by comparing the analytical results with the numerical results obtained from the time-domain and frequency numerical methods. The uniform axial flow has different effects on the sound radiation for different types of rotating sources. For the RMPS and RADPS, the uniform axial flow causes that the maximum acoustic pressure level inclines to the upstream direction. For the results of RRDPS and RCDPS, the uniform axial flow does not affect the radiation direction of the maximum acoustic pressure level, however, it does affect the distribution of acoustic energy: the higher the flow Mach number is, the more percent of the acoustic energy is radiated into the plane of source rotation. Based on this paper, future work can be carried out to analyze the noise of the axial flow fan with considering the effect of the flow Mach number.

Acknowledgments

The authors express their heartfelt appreciation for the valuable comments of the anonymous reviewers. The research has been supported by the National Natural Science Foundation of China (No. 51476123, No. 51206127), the Research Fund for the Doctoral Program of Higher Education of China (No. 20120201120060) and Fundamental Research Funds for the Central Universities (xjj2012036).

References