Investigating the effect of hydrogen addition on cyclic variability in a natural gas spark ignition engine: Wavelet multiresolution analysis

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Abstract

Using wavelet-based multiresolution analysis, this study investigates the effect of hydrogen addition on cyclic variability in a natural gas spark ignition engine. The engine is operated at 3000 rpm, and a lean combustible mixture with excess air ratio of 1.4 is used. Three cases are examined: natural gas with no hydrogen added, and natural gas with the addition of 23% and 40% hydrogen by volume. The time series of the indicated mean effective pressure are analyzed over 192 engine cycles. The method of maximal overlap discrete wavelet transform is used to decompose the time series into five levels with different frequency bands, each level consisting of a “detail” signal and an “approximation” signal. The root mean square amplitude of the detail signal at each level is used as a measure of cyclic variability. The results reveal that with the addition of 23% hydrogen, the root mean square value of the detail signal in each of the five bands is less than that for 100% natural gas. When the amount of hydrogen addition is increased to 40%, the root mean square value in each of the five bands is further reduced. In other words, hydrogen addition has a pronounced effect on reducing the cyclic variability of the indicated mean effective pressure.

1. Introduction

In order to meet the stringent emission standards for reducing environmental pollution, alternative fuels are increasingly being used in internal combustion engines, replacing conventional fuels such as gasoline and diesel. In addition, engines are being operated with very lean combustible mixtures. An advantage of the lean burn strategy is low NO x emissions due to lower flame temperature. However, operating very close to the lean flammability limit significantly reduces thermal efficiency. Ultra lean burning may also lead to local extinction, increase in unburned hydrocarbon (UHC) and CO emissions, loss of engine stability, and large-amplitude oscillations in pressure that can result in mechanical damage [1]. An alternate way to reduce NO x emission is to retard the spark timing; but this also causes a decrease in thermal efficiency and increases UHC emissions. By virtue of its abundant availability, natural gas is one of the alternative fuels which is used most extensively [2]. The main constituent of natural gas is methane. Methane has a lower carbon-to-hydrogen ratio than gasoline or diesel, which results in lower UHC and CO emissions. Under lean burn operation, a natural gas engine also produces low NO x emissions. However, because of relatively low flame velocity of methane, a lean burn natural gas engine has low thermal efficiency. It has been shown that addition of hydrogen to natural gas can extend the lean flammability limit and also increase the flame velocity, thus improving thermal efficiency and maintaining combustion stability under lean burn conditions [3].

Hydrogen has many desirable properties as a supplemental fuel that can improve the performance of a natural gas engine. The flame velocity of hydrogen is much higher than methane; therefore, the flame velocity of the combustible mixture will be substantially increased by the addition of hydrogen to natural gas, and this will result in oxidation with less heat transfer to the surroundings. In addition, the quenching gap of hydrogen is less than methane and thus hydrogen addition will allow the fuel to travel closer to the cylinder wall and farther into the crevices before being extinguished, thereby promoting complete combustion. These two factors will improve the thermal efficiency of the engine. Compared to methane, hydrogen can burn in mixtures that are five times leaner. As a consequence, hydrogen addition can extend the lean flammability limit allowing combustion under leaner conditions and achieving lower exhaust emissions. In particular, hydrogen addition reduces the formation of CO, CO 2 and unburned hydrocarbons. However, due to the fact that the flame temperature of hydrogen is higher than methane, NO x formation is increased; but research has shown that by retarding the spark timing, NO x emission can be reduced below its value with only natural gas without adversely affecting thermal efficiency [4]. Finally, because hydrogen has higher diffusivity and lower density than methane,
hydrogen-premixing will make it much easier to form a homogenous mixture; this will improve burning, which will in turn increase the power output of the engine. Several investigators have examined the effect of hydrogen addition on exhaust emissions and thermal efficiency of natural gas engines [5–18].

It is well known that the process variables such as pressure in an internal combustion engine undergo cycle-to-cycle variations (CCV) [19,20]. The CCV can become severe under lean burn conditions or for a highly dilute mixture with exhaust gas recirculation (EGR), and thus limit the potential benefits that can be derived from these operating modes. Cycle-to-cycle variations have been observed in spark ignition, compression ignition and homogeneous charge compression ignition engines [21–29]. In conventional spark ignition engines, lean-burn operation may bring about a series of problems, among which excessive cyclic combustion variations is the most prominent one [27]. The CCV may reduce fuel efficiency, lead to operational instabilities and cause knocking or misfire. These variations may also result in unwanted engine vibrations and noise, and produce excessive emissions of unburned hydrocarbons. It has been estimated that elimination of the CCV may result in about 10% increase in power output for the same fuel consumption in a gasoline engine [30]. Cycle-to-cycle variations have been observed in natural gas engines under different operating conditions [31–42]. Due to the fact that methane has high C–H bond energy, it leads to high ignition temperature and low burning velocity, resulting in large cycle-to-cycle variations (CCV) especially under lean burn conditions. It has been suggested that addition of hydrogen can be an effective way of reducing the CCV. As mentioned above, the flame velocity of hydrogen is much higher than methane; thus the flame velocity of the combustible mixture will be increased by hydrogen addition, and this will reduce the CCV. Furthermore, the minimum ignition energy of hydrogen is much less than that of natural gas; as a consequence, hydrogen addition may also lead to a reduction of CCV, especially in the initial stages of flame development.

Among the various investigators [31–42], Ma et al. [40] and Wang et al. [41] examined the effect of hydrogen addition to natural gas on the CCV of the indicated mean effective pressure (IMEP). The IMEP is the average pressure in the combustion chamber over one engine cycle. In particular, by adding 20% volume fraction of hydrogen, Ma et al. [40] demonstrated that hydrogen addition can reduce the CCV in a lean burn natural gas spark ignition engine. Wang et al. [41] added different volume fractions of hydrogen to natural gas and showed that increasing the amount of hydrogen under lean burn conditions can progressively reduce the CCV. Like many others, Ma et al. [40] and Wang et al. [41] used the coefficient of variation (COV) of the IMEP time series as a measure of the CCV.

For a time series \( \{x_i\}, i = 1, 2, 3, \ldots, N \), the COV is defined as the ratio of its standard deviation \( (\sigma) \) to its mean value \( (\mu) \), and is usually expressed in percent form.

\[
\text{COV} = \frac{\sigma}{\mu} \times 100\%,
\]

where

\[
\mu = \frac{1}{N} \sum_{i=1}^{N} x_i, \quad \sigma = \left[ \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \right]^{1/2}.
\]

The COV is a useful statistic for comparing the degree of variation between two time series even when their mean values are quite different from each other. A higher (lower) value of COV indicates larger (smaller) CCV. For the IMEP time series, the COV provides a single overall numerical measure of the CCV characterizing the temporal variability in the data. However, it does not reflect the spectral characteristics of the time series. One of the objectives of this paper is to develop a wavelet-based spectral-temporal approach to describe the CCV of the IMEP time series. This is done by performing a multiresolution analysis (MRA) of the time series using the Maximal Overlap Discrete Wavelet Transform (MODWT) [43]. The MRA is an additive decomposition which breaks up a time series into a number of “details” and a single “smooth”. Each detail is a time series describing variations on a particular timescale, whereas the smooth describes the low frequency variations. A second objective of this paper is to introduce a different measure of CCV, based on MODWT, and use it to estimate the effect of hydrogen addition on cyclic variability of IMEP in a natural gas spark ignition engine. The root mean square (rms) amplitude of each of the detail signals derived from MRA of the IMEP time series is used as the new measure of CCV. By virtue of the fact that the MRA decomposes the time series into different frequency bands, it provides an improved representation of the cyclic variability than a single numerical measure such as the COV which is based only on temporal variations. MODWT is a variant of the more conventional Discrete Wavelet Transform (DWT), with certain additional advantages (see Appendix A). MRA based on both MODWT and DWT have been used for time series analysis in a wide variety of applications [44–52]. However, to our knowledge, these techniques have not been applied to investigate the cycle-to-cycle variations in internal combustion engines.

The presentation in the remainder of this paper is organized as follows. The experimental procedure is described briefly in Section 2, and the wavelet-based multiresolution analysis methodology is described in the Appendix. In Section 3, we present and discuss the results of applying the MRA to the IMEP time series of a natural gas engine with and without hydrogen addition. Finally, in Section 4, a few concluding remarks are given.

### 2. Experimental procedure

The natural gas spark ignition engine used in this study was modified from a HH368Q three-cylinder gasoline engine. The engine specifications are listed in Table 1. The natural gas–hydrogen supply system consisted of a fuel tank, a pressure regulator, and a gas mixer. A step motor was used to regulate the amount of natural gas–hydrogen mixture and the excess air ratio. The excess air ratio is the ratio of actual air–fuel ratio to the stoichiometric air–fuel ratio. An electronic control unit (ECU) was designed to control the spark timing and the step motor. Hydrogen used in this experiment had a purity of 99.995%, and the natural gas had the composition given in Table 2. The natural gas and hydrogen were mixed in desired volumetric proportions in the fuel tank, and the mixture was

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Engine specifications.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engine type</td>
<td>HH368Q gasoline engine</td>
</tr>
<tr>
<td>Bore (mm)</td>
<td>68.5</td>
</tr>
<tr>
<td>Stroke (mm × mm)</td>
<td>72</td>
</tr>
<tr>
<td>Displacement (ml)</td>
<td>796</td>
</tr>
<tr>
<td>Compression ratio</td>
<td>9.4</td>
</tr>
<tr>
<td>Ignition sequence</td>
<td>1–3–2</td>
</tr>
<tr>
<td>Rated speed (rpm)</td>
<td>5500</td>
</tr>
<tr>
<td>Rated power (kW)</td>
<td>26.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Composition of natural gas.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constituent</td>
<td>CH4</td>
</tr>
<tr>
<td>Volume fraction (%)</td>
<td>96.16</td>
</tr>
</tbody>
</table>
brought to atmospheric pressure to mix with air through a venturi gas mixer before being delivered to the intake port. A Horiba MEXA-700k gas analyzer measured the excess air ratio and the mixture concentration. The engine was operated at 3000 rpm with a brake mean effective pressure (BMEP) of 0.16 MPa, and a lean mixture with excess air ratio of 1.4 was used. The experiments were conducted at wide open throttle (WOT), and the ignition timing was adjusted to the maximum brake torque (MBT) value. The in-cylinder pressure was measured by a Kistler 6117BF17 piezoelectric pressure transducer with a resolution of 10 Pa, and the dynamic top dead center (TDC) was calibrated by motoring. The crank angle information was obtained by means of a crank angle encoder (Kistler Model 2613B) mounted on the main shaft. The pressure signal was recorded for every 0.1 degree of the crank angle. The crank angle encoder (Kistler Model 2613B) mounted on the main shaft. The pressure and crank angle signals were acquired using the Yokogawa DL750 data acquisition system. In order to analyze the cycle-to-cycle variations (CCV), the pressure measurements were made for 192 consecutive engine cycles. Further details of the experimental procedure can be found in the paper by Liu et al. [18].

From the measured pressure, the IMEP which is the average pressure in the cylinder over one engine cycle, is calculated as follows.

\[
\text{IMEP} = \frac{W_c}{V_d},
\]

where \(V_d\) is the engine displacement volume, and \(W_c\) is the amount of work done per cycle:

\[
W_c = \int P dV,
\]

\(P\) being the actual (i.e., measured) pressure in the cylinder. Experiments were performed for the following three cases: (a) natural gas with no hydrogen added, (b) natural gas with 23% hydrogen added by volume, and (c) natural gas with the addition of 40% hydrogen by volume.

### Table 3

Periodicities of the detail components.

<table>
<thead>
<tr>
<th>Detail component</th>
<th>Period (cycle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_1)</td>
<td>2–4</td>
</tr>
<tr>
<td>(D_2)</td>
<td>4–8</td>
</tr>
<tr>
<td>(D_3)</td>
<td>8–16</td>
</tr>
<tr>
<td>(D_4)</td>
<td>16–32</td>
</tr>
<tr>
<td>(D_5)</td>
<td>32–64</td>
</tr>
</tbody>
</table>

### Table 4

Root mean square (rms) values of the detail components of the IMEP time series for different natural gas–hydrogen blends.

<table>
<thead>
<tr>
<th>Wavelet</th>
<th>Detail</th>
<th>No H(_2) added</th>
<th>23% H(_2) added</th>
<th>40% H(_2) added</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sym8</td>
<td>(D_1)</td>
<td>0.0639</td>
<td>0.0225</td>
<td>0.0135</td>
</tr>
<tr>
<td></td>
<td>(D_2)</td>
<td>0.0629</td>
<td>0.0161</td>
<td>0.0091</td>
</tr>
<tr>
<td></td>
<td>(D_3)</td>
<td>0.0369</td>
<td>0.0062</td>
<td>0.0040</td>
</tr>
<tr>
<td></td>
<td>(D_4)</td>
<td>0.0240</td>
<td>0.0062</td>
<td>0.0037</td>
</tr>
<tr>
<td></td>
<td>(D_5)</td>
<td>0.0653</td>
<td>0.0229</td>
<td>0.0136</td>
</tr>
<tr>
<td></td>
<td>(D_6)</td>
<td>0.0662</td>
<td>0.0168</td>
<td>0.0096</td>
</tr>
<tr>
<td>Sym16</td>
<td>(D_1)</td>
<td>0.0501</td>
<td>0.0143</td>
<td>0.0086</td>
</tr>
<tr>
<td></td>
<td>(D_2)</td>
<td>0.0501</td>
<td>0.0143</td>
<td>0.0086</td>
</tr>
<tr>
<td></td>
<td>(D_3)</td>
<td>0.0395</td>
<td>0.0085</td>
<td>0.0043</td>
</tr>
<tr>
<td></td>
<td>(D_4)</td>
<td>0.0248</td>
<td>0.0064</td>
<td>0.0038</td>
</tr>
<tr>
<td></td>
<td>(D_5)</td>
<td>0.0640</td>
<td>0.0225</td>
<td>0.0135</td>
</tr>
<tr>
<td></td>
<td>(D_6)</td>
<td>0.0634</td>
<td>0.0162</td>
<td>0.0091</td>
</tr>
<tr>
<td>Coif4</td>
<td>(D_1)</td>
<td>0.0489</td>
<td>0.0136</td>
<td>0.0065</td>
</tr>
<tr>
<td></td>
<td>(D_2)</td>
<td>0.0373</td>
<td>0.0062</td>
<td>0.0041</td>
</tr>
<tr>
<td></td>
<td>(D_3)</td>
<td>0.0248</td>
<td>0.0062</td>
<td>0.0037</td>
</tr>
<tr>
<td></td>
<td>(D_4)</td>
<td>0.0636</td>
<td>0.0223</td>
<td>0.0135</td>
</tr>
<tr>
<td></td>
<td>(D_5)</td>
<td>0.0621</td>
<td>0.0159</td>
<td>0.0090</td>
</tr>
<tr>
<td></td>
<td>(D_6)</td>
<td>0.0483</td>
<td>0.0133</td>
<td>0.0064</td>
</tr>
<tr>
<td>Bior6.8</td>
<td>(D_1)</td>
<td>0.0364</td>
<td>0.0061</td>
<td>0.0040</td>
</tr>
<tr>
<td></td>
<td>(D_2)</td>
<td>0.0238</td>
<td>0.0061</td>
<td>0.0036</td>
</tr>
</tbody>
</table>

### 3. Results and discussion

Using the MODWT-based multiresolution analysis described in the Appendix A, each of the three IMEP time series for the natural gas engine (with and without hydrogen addition) is decomposed into five levels, each level consisting of an approximation component and a detail component. As mentioned in the Introduction,
the detail component at each level represents the high frequency fluctuations, whereas the approximation describes the low frequency variations. With this 5-level decomposition, each IMEP time series can be represented as follows (see Eq. (A10) in the Appendix A).

\[ S = D_1 + D_2 + D_3 + D_4 + A_5, \]

where \( D_j \), \( j = 1–5 \) denote the detail signal at levels 1 through 5, and \( A_5 \) is the approximation signal at level 5. The details \( D_1, D_2, D_3, D_4 \) and \( D_5 \) correspond to the frequency bands of 2–4 cycle, 4–8 cycle, 8–16 cycle, 16–32 cycle, and 32–64 cycle, respectively (see Table 3).

The decomposition given in Eq. (5) is shown in a hierarchical fashion in Fig. 1. In our computations, we have used the following wavelet basis functions: sym8, sym16, db8, db14, coif4, and bior6.8. The abbreviations sym, db, coif and bior stand for symmlet, Daubechies, coiflet, and biorthogonal, respectively. These wavelet functions have been used effectively for multiresolution analysis in many applications [44–52].

Consider first the spark ignition engine fueled by 100% natural gas. The IMEP time series, the detail components (\( D_1 – D_5 \)), and the approximation (\( A_5 \)) of the five-level decomposition for this case are shown in Fig. 2. This figure illustrates the quasiperiodic nature of the CCV of the IMEP time series in the various frequency bands. The rms values of the detail components in these five levels obtained using sym8, sym16, db8, db14, coif4, and bior6.8 as wavelets are listed in Table 4. When db8 and db14 are used as wavelets, the results were found to be identical to those obtained with sym8 and sym16, respectively; for the sake of brevity, they are not included in Table 4. Next we consider the engine fueled by natural gas with the addition of 23% and 40% hydrogen by volume. Figs. 3 and 4 depict the IMEP time series, the detail components (\( D_1 – D_5 \)), and the approximation (\( A_5 \)) of the five-level decomposition for these two cases. The rms values of the detail components in the five levels obtained using sym8, sym16, coif4, bior6.8 as wavelet functions are given in Table 4. From a visual inspection of the detail components, \( D_1 – D_5 \), shown in Figs. 2 and 3, it is apparent that with the addition of 23% hydrogen, each detail component becomes smaller compared to when no hydrogen was added. As the volume fraction of hydrogen is increased to 40%, Figs. 3 and 4 indicate that the corresponding detail components are further reduced. In fact, with the addition of 40% hydrogen, the reduction is quite significant from the engine with no hydrogen added. The bar graph in Fig. 5 depicts the rms amplitudes of the details, \( D_1 – D_5 \), in all three cases. The following trends can be observed from the bar graph (see also Table 4): (i) the rms amplitudes in each of these five bands are smaller for the hydrogen-added engine than the natural gas fueled engine, (ii) as the amount of hydrogen added increases, the rms amplitudes in each band progressively decrease, (iii) in each of the three cases, the rms amplitude decreases from \( D_1 \) to \( D_5 \), i.e., as the frequency bandwidth of the CCV decreases; in other words, the CCV are diminished at lower frequencies. In addition, we note that even though the various wavelet basis functions we have used for MODWT are different in shape, the rms amplitudes are not sensitive to the choice of the specific wavelet function (see Table 4).

In Table 5, the \( R_{23} \) values show the ratio (percent) of the rms amplitudes of the corresponding detail components, \( D_1 – D_5 \), for the natural gas engine with 23% hydrogen added to no hydrogen added. Similar results for the case of 40% hydrogen addition are given by the \( R_{40} \) values in this Table. Each ratio is calculated using the rms values presented in Table 4 obtained with sym8 wavelet. Table 5 reveals that with 40% hydrogen addition, the rms amplitudes of all the detail components, \( D_1 – D_5 \), are significantly smaller for the hydrogen-added engine than the natural gas fueled engine.

![Fig. 3](image-url) The IMEP time series, the detail signals, \( D_1 – D_5 \), and the approximation signal, \( A_5 \), for the engine fueled by natural gas mixed with 23% hydrogen. The wavelet function used is sym8. Engine speed = 3000 rpm, BMEP = 0.16 MPa, and excess air ratio = 1.4.

![Fig. 4](image-url) The IMEP time series, the detail signals, \( D_1 – D_5 \), and the approximation signal, \( A_5 \), for the engine fueled by natural gas mixed with 40% hydrogen. The wavelet function used is sym8. Engine speed = 3000 rpm, BMEP = 0.16 MPa, and excess air ratio = 1.4.
reduced, and the largest reduction occurs in \( D_4 \), which is in the 16–32 cycle periodic band.

### 4. Concluding remarks

Using MODWT-based multiresolution analysis we have investigated the effect of hydrogen addition on the cycle-to-cycle variations (CCV) of the indicated mean effective pressure (IMEP) in a natural gas spark ignition engine. Three cases were examined: natural gas with no hydrogen added, and natural gas blended with 23% and 40% hydrogen by volume. Based on the results obtained, the following conclusions can be made.

1. Hydrogen addition can cause a significant reduction in the cycle-to-cycle variations of the indicated mean effective pressure.
2. The main advantage of this multiresolution analysis technique is that, based on the rms values of the detail components in different frequency bands, it provides a simple but detailed quantitative procedure to estimate the effect of hydrogen addition on cyclic variability in a natural gas spark ignition engine.

Finally we note that in this paper, we have analyzed the various IMEP time series over 192 engine cycles. By extending the analysis over a larger number of cycles, the CCV at lower frequencies can be analyzed. Such analyses may be useful for understanding the effect of hydrogen addition on the long-term dynamics of the cyclic variability in natural gas spark ignition engines.

### Appendix A

As mentioned in Section 1, the maximal overlap discrete wavelet transform (MODWT) is a variant of the discrete wavelet transform (DWT). Both DWT and MODWT may be considered as special band pass filtering techniques that decompose a signal into different levels with different frequency bands. In order to describe the MODWT, it is appropriate to first describe the DWT. Consider a signal given by a discrete time sequence \( x(n) \). The DWT of this signal, based on a dyadic grid of scale and time, is given by the coefficients [53]

\[
C_{jk} = 2^{-j/2} \sum_n x(n) \Psi_{jk}(n).
\]  

(A1)

Here \( \Psi(n) \) is a wavelet basis function, and

\[
\Psi_{jk}(n) = 2^{-j/2} \Psi(2^{-j} n - k)
\]

(A2)

is the scaled and translated version of \( \Psi(n) \). The symbols \( j \) and \( k \) are the indices for the scaling coefficient and translation coefficient, respectively. The DWT uses two types of wavelet basis function, the mother wavelet which we will denote by \( \Phi(n) \), and the father wavelet which will be denoted by \( \Psi(n) \). The mother wavelet operates as a band pass filter, whereas the father wavelet acts as a low pass filter. The DWT decomposes a signal into a set of approximation coefficients and a set of detail coefficients. The approximation coefficients capture the large scale or low frequency features of the signal, whereas the detail coefficients reveal the small scale or high frequency variations.

For the discrete-time signal \( x(n) \), the approximation and detail coefficients at the \( j \)-th level are found from:

\[
\alpha_{jk} = 2^{-j/2} \sum_n x(n) \Phi(2^{-j} n - k),
\]  

(A3)

\[
\beta_{jk} = 2^{-j/2} \sum_n x(n) \Psi(2^{-j} n - k),
\]  

(A4)

where the functions \( \Phi(n) \) and \( \Psi(n) \) are the scaled and translated versions of the father and mother wavelets, \( \Phi(n) \) and \( \Psi(n) \), respectively, similar to Eq. (A2).

\[
\phi_{jk}(n) = 2^{-j/2} \phi(2^{-j} n - k),
\]  

(A5)

\[
\psi_{jk}(n) = 2^{-j/2} \psi(2^{-j} n - k).
\]

(A6)

The approximation signal and the detail signal at the \( j \)-th level are obtained as

\[
h_j(n) = \sum_{k=-\infty}^{\infty} \alpha_{jk} \phi_{jk}(n),
\]  

(A7)

\[
g_j(n) = \sum_{k=-\infty}^{\infty} \beta_{jk} \psi_{jk}(n).
\]  

(A8)

By computing the approximation and detail signals at level \( J \), the original signal \( x(n) \) can be reconstructed as follows.

\[
x(n) = \sum_{k=-\infty}^{\infty} \alpha_{j,k} \phi_{j,k}(n) + \sum_{j=1}^{J} \sum_{k=-\infty}^{\infty} \beta_{j,k} \psi_{j,k}(n)
\]  

(A9)

### Table 5

<table>
<thead>
<tr>
<th>Detail</th>
<th>( R_{23} ) (%)</th>
<th>( R_{24} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>35.2</td>
<td>21.1</td>
</tr>
<tr>
<td>( D_2 )</td>
<td>25.6</td>
<td>14.5</td>
</tr>
<tr>
<td>( D_3 )</td>
<td>27.7</td>
<td>13.3</td>
</tr>
<tr>
<td>( D_4 )</td>
<td>16.8</td>
<td>10.8</td>
</tr>
<tr>
<td>( D_5 )</td>
<td>25.8</td>
<td>15.4</td>
</tr>
</tbody>
</table>

\( R_{23} \) = ratio (%) of the rms amplitudes of the corresponding detail components for the natural gas engine with 23% hydrogen added to no hydrogen added. Each ratio is calculated using the rms values presented in Table 4 obtained with sym8 wavelet.

\( R_{24} \) = Ratio (percent) of the rms amplitudes of the corresponding detail components for the natural gas engine with 40% hydrogen added to no hydrogen added. Each ratio is calculated using the rms values presented in Table 4 obtained with sym8 wavelet.

### Fig. 5.

Bar graph showing the rms amplitudes of the detail signals, \( D_1 - D_5 \), for the three cases examined: (a) 100% natural gas, (b) natural gas with 23% hydrogen added, and (c) natural gas with 40% hydrogen added. The wavelet function used is sym8.
The additive decomposition given by Eq. (A10) is usually referred to as multiresolution analysis (MRA) [43,53].

In practice, DWT is implemented with filters using the pyramid algorithm due to Mallat [53]. The pyramid algorithm starts by filtering (convolving) the observed time series using a pair of high-pass and low-pass filters, and downsamples the filtered output from each filter by a factor of 2. This constitutes first level of decomposition. The downsampled output from the high-pass filter is referred to as the detail coefficient, whereas the downsampled output from the low-pass filter is called the approximation coefficient. Subsequently, the above filtering and downsampling operations are repeated on the downsampled output from the low-pass filter. By continuing this procedure, the original time series is decomposed into different levels, each level containing an approximation coefficient and a detail coefficient. The iterated filter bank representation of the DWT procedure is given in many works; see for example, [54].

We now briefly describe the MODWT. The MODWT is also referred to by various names such as undecimated DWT [55], stationary DWT [56], translation-invariant DWT [57], nondecimated DWT [58], time-invariant DWT [59]. It was mentioned in Section 1 that both MODWT and DWT can be used to perform multiresolution analysis of a time series. However, there are important differences between the two techniques. The DWT is applicable only to time series of dyadic length (2^p), p being a positive integer), whereas the MODWT does not require such a restriction. Note however that for MODWT, the record length N should be divisible by 2^M, M being the number of chosen decomposition levels. The DWT involves downsampling by 2 at every level of decomposition and up-sampling by 2 at every level of reconstruction of a time series, but no down- or up-sampling is performed in MODWT. The MODWT is highly redundant in that for a time series given by a vector of length N, the J-level decomposition in MODWT consists of (2^J-1) * N coefficients rather than just N as in the case of DWT. The details and smooth coefficients of a MODWT multiresolution analysis are associated with zero phase filters; this means that events in the original time series are aligned with the events in the multiresolution analysis. In addition, MODWT is shift-invariant in the sense that shifting the time series by an integer amount will shift the MODWT wavelet and scaling coefficients by the same amount. This property does not hold for the DWT.

References


